Problem of the Month
Problem 8: May 2021

Problem Try these three geometry problems! Problems (a), (b), and (c) are not intended to be related to each other. In each part, a diagram is provided to give an example of a figure satisfying the conditions to be explored in that part.

(a) In trapezoid $ABCD$, $\angle BAD = 90^\circ$ and $BC$ is parallel to $AD$ with $BC < AD$. The diagonals $AC$ and $BD$ intersect at point $X$. A line parallel to $AD$ is drawn through $X$ and intersects $AB$ at $L$ and $CD$ and $M$. Determine the length of $LM$ in terms of the lengths of $BC$ and $AD$.

![Diagram of trapezoid with additional lines](image1)

(b) Suppose quadrilateral $ABCD$ has no pair of parallel sides and is inscribed in a circle. $AB$ and $DC$ are extended to meet at point $E$ and $AD$ and $BC$ are extended to meet at point $F$. The degree measures of $\angle AFB$, $\angle AED$, and $\angle EAF$ form an increasing arithmetic sequence in that order. The degree measure of each of these three angles is an integer. Find all possible values of $\angle AFB$.

![Diagram of quadrilateral inscribed in a circle](image2)

(c) Rectangle $DBCA$ has $E$ on $BC$ and $F$ on $AC$ so that $\triangle DEF$ is equilateral. Find all possible values of $\frac{BD}{AD}$.

![Diagram of rectangle with additional lines](image3)
Hint

(a) The assumption that $\angle BAD = 90^\circ$ is not needed, but it might make a few calculations or observations easier. Try to find some similar triangles. Keep in mind that corresponding altitudes of similar triangles are in the same ratio as their sides.

(b) A quadrilateral that can be inscribed in a circle, such as $ABCD$, is called a cyclic quadrilateral. It might be useful to look up a few facts about cyclic quadrilaterals. Using these facts, try to show that $\angle AFB + \angle AED + 2\angle BAD = 180^\circ$.

(c) $\triangle DEF$ being equilateral tells us that $\angle EDF = 60^\circ$ and $DE = DF$. Trigonometry might be useful in this problem. Since the question asks for the ratio $\frac{BD}{AD}$, you can assume that $AD = 1$ (or has some other fixed value) and explore the possible values of $BD$. 