Problem of the Month
Solution to Problem 6: March 2021

(a) Since there are 5 dots, each dot is connected to \(5 - 1 = 4\) dots. This gives a total of \(\frac{5 \times 4}{2} = 10\) lines where the division by 2 is because the product \(5 \times 4\) counts each line twice.

Suppose \(r\) is the number of red dots and \(b\) is the number of blue dots. Since \(n = 5\) and every dot must be coloured, \(r + b = 5\). Only the lines connecting a blue dot to a red dot are coloured blue, and since each blue dot is connected to each red dot, there are exactly \(rb\) blue lines. If a colouring is balanced, then the number of blue lines would be \(\frac{10}{2} = 5\). Thus, if the colouring is balanced, then \(rb = 5\). Since \(r\) and \(b\) are nonnegative integers, this means \(r = 1\) and \(b = 5\) or \(r = 5\) and \(b = 1\). In each case, \(r + b \neq 5\), so there can be no balanced colouring when \(n = 5\).

(b) Similar to the argument in part (a), the number of lines when \(n = 9\) is \(\frac{9 \times 8}{2} = 36\). As well, if we let \(r\) be the number of red dots and \(b\) be the number of blue dots, then the number of blue lines is \(rb\). For a colouring to be balanced, we need \(rb = 18\). Therefore, we are looking for nonnegative integers \(r\) and \(b\) such that \(rb = 18\) and \(r + b = 9\).

Since \(rb = 18\) and both \(r\) and \(b\) are nonzero, we have \(b = \frac{18}{r}\). Substituting this expression into \(r + b = 9\) gives \(r + \frac{18}{r} = 9\). Multiplying through by \(r\) and rearranging gives \(r^2 - 9r + 18 = 0\), which can be factored to get \((r - 6)(r - 3) = 0\). Therefore, the number of red dots must be either 3 or 6.

If \(r = 3\), then \(b = 9 - 3 = 6\), so the number of blue lines is \(3 \times 6 = 18\). This means the number of red lines is \(36 - 18 = 18\). If \(r = 6\), then \(b = 9 - 6 = 3\), so the number of blue lines is \(6 \times 3 = 18\) and the number of red lines is \(36 - 18 = 18\) as well.

Thus, colouring the dots so that 3 are red or 6 are red gives a balanced colouring, and there are no other possibilities.

(c) When there are \(n\) dots, there are \(\frac{n(n - 1)}{2}\) lines. Once again, we set \(r\) to be the number of red dots and \(b\) to be the number of blue dots, so that \(rb = \frac{1}{2} \times \frac{n(n - 1)}{2} = \frac{n(n - 1)}{4}\) is the number of blue lines in a balanced colouring.

Therefore, we wish to find all integers \(n > 1\) for which there are nonnegative integers \(r\) and \(b\) satisfying \(r + b = n\) and \(rb = \frac{n(n - 1)}{4}\).

By squaring both sides of the equation \(r + b = n\), we obtain \(r^2 + 2rb + b^2 = n^2\). Multiplying both sides of the equation \(rb = \frac{n(n - 1)}{4}\) by 4 gives \(4rb = n^2 - n\). Subtracting this equation from \(r^2 + 2rb + b^2 = n^2\) gives \(r^2 - 2rb + b^2 = n\) which factors as \((r - b)^2 = n\). Therefore, if there is a balanced colouring, then \(n\) must be a perfect square.

To finish the argument, we will show that if \(n\) is a perfect square, then there is a balanced
colouring. To get an idea how to do this, let us suppose \( n = m^2 \) for some positive integer \( m \). We know that if a balanced colouring exists, then \((r - b)^2 = m^2\). If \( r > b \), then \( r - b = m \). Adding this to \( r + b = n \) and dividing by 2, we have \( r = \frac{n + m}{2} \). It then follows that \( b = \frac{n - m}{2} \). If \( r < b \), we get that \( r = \frac{n - m}{2} \) and \( b = \frac{n + m}{2} \). This shows that if a balanced colouring exists, then we must have that \( r = \frac{n \pm \sqrt{n}}{2} \).

If \( n \) is a perfect square, then we can let \( r = \frac{n + \sqrt{n}}{2} \) and \( b = \frac{n - \sqrt{n}}{2} \) which gives

\[
rb = \frac{n^2 - \sqrt{n}^2}{4} = \frac{1}{2} \times \frac{n^2 - n}{2}
\]

and so the colouring is balanced. A nearly identical calculation shows that we can let \( r = \frac{n - \sqrt{n}}{2} \) and \( b = \frac{n + \sqrt{n}}{2} \) and we would also get a balanced colouring.

Therefore, there is a balanced colouring exactly when \( n \) is a perfect square. Moreover, if \( n \) is a perfect square, then the number of red dots in a balanced colouring must be either \( \frac{n + \sqrt{n}}{2} \) or \( \frac{n - \sqrt{n}}{2} \). We point out that when \( n \) is a perfect square, \( n \) and \( \sqrt{n} \) are either both even or both odd. This means the numerators \( n + \sqrt{n} \) and \( n - \sqrt{n} \) are even, so the numbers of red dots given above are both integers.

(d) Suppose \( r \) is the number of red dots, \( b \) is the number of blue dots, and \( g \) is the number of green dots. The number of lines is \( \frac{n(n - 1)}{2} \), so in a balanced colouring, we need to have \( \frac{n(n - 1)}{6} \) lines of each colour.

The lines that are coloured red are the lines connecting two red dots or the lines connecting a blue dot to a green dot. The number of lines connecting red dots to red dots is \( \frac{r(r - 1)}{2} \). This is because each of the \( r \) red dots is connected to the \( r - 1 \) other red dots, so \( r(r - 1) \) counts each such line twice. The number of lines connecting blue dots to green dots is \( bg \). Therefore, the number of red lines is

\[
\frac{r(r - 1)}{2} + bg
\]

Similar reasoning shows that the number of blue lines is

\[
\frac{g(g - 1)}{2} + rb
\]

and that the number of green lines is

\[
\frac{b(b - 1)}{2} + rg.
\]

Suppose the numbers of red lines, blue lines, and green lines are all equal. In particular, the number of red lines equals the number of blue lines, so

\[
\frac{r(r - 1)}{2} + bg = \frac{g(g - 1)}{2} + rb.
\]
Multiplying this equation by 2 and expanding gives \( r^2 - r + 2bg = g^2 - g + 2rb \). Rearranging this, we get
\[
(r^2 - g^2) + (g - r) + (2bg - 2rb) = 0
\]
which has a common factor of \( r - g \) and can be rewritten as
\[
(r - g)(r + g - 1 - 2b) = 0. \tag{1}
\]

Similarly, the number of blue lines equals the number of green lines, so
\[
g(g - 1) \frac{2}{2} + rb = \frac{b(b - 1)}{2} + rg
\]
which is equivalent to
\[
g^2 - b^2 + b - g + 2rb - 2rg = 0
\]
and then factored as
\[
(g - b)(g + b - 1 - 2r) = 0. \tag{2}
\]
Equating the number of red lines and the number of green lines gives
\[
\frac{r(r - 1)}{2} + bg = \frac{b(b - 1)}{2} + rg,
\]
which is equivalent to
\[
(r - b)(r + b - 1 - 2g) = 0. \tag{3}
\]

Suppose \( r, b, \) and \( g \) are three distinct integers. This means \( r - g \neq 0 \), so Equation (1) implies \( r + g - 1 - 2b = 0 \) or \( r + g = 1 + 2b \). Similarly, \( g - b \neq 0 \) and \( r - b \neq 0 \), so Equations (2) and (3) imply \( b + g = 1 + 2r \) and \( r + b = 1 + 2g \), respectively. Adding these three equations, we get
\[
(r + g) + (b + g) + (r + b) = (1 + 2b) + (1 + 2r) + (1 + 2g)
\]
which implies \( 2(r + b + g) = 3 + 2(r + b + g) \) so \( 3 = 0 \). Of course, this is not true, which means \( r, b, \) and \( g \) cannot be three distinct integers. In other words, at least two of \( r, b, \) and \( g \) are equal to each other.

One way for Equations (1), (2), and (3) to be satisfied simultaneously is when \( r = b = g \). This implies that there is some positive integer \( k \) with \( n = 3k \) and \( r = b = g = k \).

Otherwise, there are three possibilities: \( r = b \) with \( g \) different from \( r \) and \( b \), \( r = g \) with \( b \) different from \( r \) and \( g \), and \( b = g \) with \( r \) different from \( b \) and \( g \).

If \( r = b \) and \( g \) is different from \( r \) and \( b \), then \( r \neq g \), so Equation (1) implies \( r + g - 1 - 2b = 0 \). Substituting \( r = b \), we get \( g - 1 - b = 0 \) or \( g = 1 + b \). This shows that \( g \) is one more than the common value of \( r \) and \( b \). Thus, there is some positive integer \( k \) so that \( n = 3k + 1 \) and \( r = b = k \) and \( g = k + 1 \).

Similar analysis shows that if \( r = g \) with \( b \) different from \( r \) and \( b \), then \( n = 3k + 1 \) for some positive integer \( k \) and \( r = g = k \) with \( b = k + 1 \). As well, if \( b = g \) and \( r \) is different form \( b \) and \( g \), then \( n = 3k + 1 \) with \( b = g = k \) and \( r = k + 1 \).

We have now argued that if there is a balanced colouring of \( n \) dots, then one of these two statements must be true:

- \( n = 3k \) for some positive integer \( k \) and there are \( k \) dots of each colour.
• $n = 3k + 1$ for some positive integer $k$ and $(r, b, g)$ is one of $(k, k, k + 1)$, $(k, k + 1, k)$, and $(k + 1, k, k)$.

To finish the solution, we will check that the colouring is indeed balanced in each of the four situations described.

If $n = 3k$ for some positive integer $k$ and $r = b = g = k$, then the number of red lines is

$$\frac{k(k - 1)}{2} + k^2 = \frac{k^2 - k}{2} + k^2$$

$$= \frac{3k^2 - k}{2}$$

$$= \frac{9k^2 - 3k}{6}$$

$$= \frac{3k(3k - 1)}{6}$$

$$= \frac{n(n - 1)}{6}.$$  

A similar calculation shows that there are the same number of blue and green lines. Therefore, there is a balanced colouring when $n = 3k$ for some positive integer $k$.

If $n = 3k + 1$ for some positive integer $k$ and $r = b = k$ with $g = k + 1$, then the number of red lines is

$$\frac{k(k - 1)}{2} + k(k + 1) = \frac{k^2 - k + 2k^2 + 2k}{2}$$

$$= \frac{3k^2 + k}{2}$$

$$= \frac{9k^2 + 3k}{6}$$

$$= \frac{3k(3k + 1)}{6}$$

$$= \frac{(n - 1)n}{6}.$$  

The number of green lines is also $\frac{n(n - 1)}{6}$ by essentially an identical calculation, and the number of blue lines is

$$\frac{(k + 1)k}{2} + k^2 = \frac{3k^2 + k}{2}$$

$$= \frac{n(n - 1)}{6}.$$  

by another similar calculation. The situations when $r = g = k$ and $b = k + 1$ and $b = g = k$ and $r = k + 1$ can be handled similarly.

We have now shown that when $n = 3k$ or $n = 3k + 1$, there is a balanced colouring. Therefore, a balanced colouring exists exactly when $n = 3k$ or $n = 3k + 1$ for some positive integer $k$. 
Further Remark

If you are familiar with modular arithmetic, then you might have noticed a pattern in the rules for colouring the lines. In the first three parts, we could have said that each point is “coloured” by either 0 or 1, and the colour of a line is the sum modulo 2 of the “colours” of the dots it connects. “Modulo 2” means the remainder after division by 2. So in this system of arithmetic, $0 + 0 = 0$, $0 + 1 = 1 + 0 = 1$, and $1 + 1 = 0$ since the remainder after dividing 2 by 2 is 0. In our problem, red corresponds to 0 and blue corresponds to 1. Similarly, in part (d) we might instead label each dot by 0, 1, or 2 (red is 0, blue is 1, and green is 2). The label is the sum modulo 3, the remainder after division by 3. This means $0 + 0 = 0$, $0 + 1 = 1 + 0 = 1$, $0 + 2 = 2 + 0 = 2$, $1 + 2 = 2 + 1 = 0$ (since the remainder after dividing 3 by 3 is 0), and $2 + 2 = 1$ (since the remainder after dividing 4 by 3 is 1).

With this in mind, we can imagine labelling the dots by the integers 0, 1, 2, or 3 and labelling the lines by the sum modulo 4, or even more generally, we could label dots label by 0, 1, 2, 3, and so on up to $n - 1$ and label the lines by the sum modulo $n$. What should the definition of “balanced” be in general, and what can you say about balanced colourings in general?