Problem of the Month
Problem 6: March 2021

Here is a simple activity that leads to an interesting math problem.

• For a positive integer \( n > 1 \), draw \( n \) dots on a piece of paper. Draw a line to connect each pair of dots. The lines do not need to be straight, but should be drawn so that they do not pass through any dots other than the two they connect. If two lines intersect, the intersection does not define a new dot.

• Colour each dot either red or blue in any way that you like.

• Colour each line as follows: If the line connects two dots of the same colour, colour the line red. Otherwise, colour the line blue.

Call a colouring of the dots balanced if it leads to the lines being coloured so that there is the same number of blue lines as red lines.

(a) Show that there is no balanced colouring when \( n = 5 \).

(b) Show that there is a balanced colouring when \( n = 9 \). Find all possibilities for the number of red dots in a balanced colouring when \( n = 9 \).

(c) Determine all \( n \) for which there is a balanced colouring. For each such \( n \), determine all possibilities for the number of red dots in a balanced colouring.

For part (d), the dots can now be coloured red, blue, or green. The table below describes how the lines should be coloured once the dots are coloured. For example, the letter \( R \) is in the cell corresponding to the row for \( B \) and the column for \( G \). This means that if a line connects one blue dot and one green dot, then it is to be coloured red.

\[
\begin{array}{ccc}
R & G & B \\
R & R & G & B \\
G & G & B & R \\
B & B & R & G \\
\end{array}
\]

For part (d), we redefine a balanced colouring of the dots to mean a colouring leads to equal numbers of red, blue, and green lines.

(d) Describe all \( n \) for which there is a balanced colouring.