Problem of the Month
Problem 5: February 2021

For an integer $n \geq 3$, we define $T_n$ to be the triangle with side lengths $n - 1$, $n$, and $n + 1$, and define $A_n$ to be the area of $T_n$. We will say that an integer $n \geq 3$ is remarkable if $A_n$ is an integer.

(a) Determine all integers $n$ for which $T_n$ is right-angled.

(b) Suppose $n$ is a remarkable integer. Prove that

(i) $\frac{n^2 - 4}{3}$ is a perfect square,

(ii) $n$ is not a multiple of 3, and

(iii) $n$ is even.

(c) There are three remarkable integers less than or equal to 100. Determine these three integers.

(d) The only remarkable integers between 100 and 10,000 are $n = 194$, $n = 724$, and $n = 2702$. Find a polynomial function $f(n)$ of degree greater than 1 with the property that if $n$ is a remarkable integer, then $f(n)$ is also a remarkable integer. Use this polynomial to deduce that there are infinitely many remarkable integers.

(e) Explain how to find all remarkable integers. This should involve somehow describing an infinite set of remarkable integers as well as justification that your set is complete. Keep in mind that the infinite set from part (d) may not include all remarkable integers.