Problem of the Month
Problem 4: January 2021

In this problem, we will explore when a quadratic polynomial of the form $x^2 + ux + v$ can be decomposed as the sum of the squares of two other polynomials. Keep in mind that a constant function is a polynomial. All polynomials in the problem statements below are assumed to have real coefficients, though they may not have real roots.

(a) Find at least three pairs $(p(x), q(x))$ of polynomials such that $(p(x))^2 + (q(x))^2 = x^2 + 2x + 2$.

(b) Suppose $f(x) = x^2 + ux + v$ has the property that $f(x) \geq 0$ for all real numbers $x$. Prove that there are polynomials $p(x)$ and $q(x)$ such that $x^2 + ux + v = (p(x))^2 + (q(x))^2$.

In the remaining parts of this problem, we will say that the pair of polynomials $(p(x), q(x))$ is special for the polynomial $x^2 + ux + v$ if

- the coefficients of $p(x)$ and $q(x)$ are all rational, and
- $x^2 + ux + v = (p(x))^2 + (q(x))^2$.

(c) Prove that there are no special pairs for $x^2 + x + 1$.

(d) Prove that if there is a special pair for $x^2 + ux + v$, then $u$ and $v$ are both rational and there is a rational number $r$ such that $4v - u^2 = r^2$.

(e) Prove that if there is a special pair for $x^2 + ux + v$, then there are infinitely many special pairs for $x^2 + ux + v$. 