Problem of the Month

Problem 1: October 2020

(a) Let $\theta$ be an angle with $0 < \theta < 45^\circ$. In the diagram, points $A$ and $B$ are configured so that $\angle AOB = 2\theta$ and $\triangle AOB$ is isosceles with $AO = BO$.

A circle is inscribed in $\triangle AOB$ and another circle is drawn so that it is tangent to the larger circle as well as $OA$ and $OB$. In terms of $\theta$, find the ratio of the radius of the larger circle to the radius of the smaller circle.

(b) Similar to part (a), an equilateral triangle has a circle inscribed in it. Three circles are then drawn, each tangent to two of the sides of the triangle as well as the larger circle. Another three circles are then drawn, each tangent to two of the three sides of the triangle as well as one of the circles drawn in the previous step.

If this process is continued indefinitely, what fraction of the area of the triangle is covered by circles?
(c) Suppose $\triangle AOB$ and $\theta$ are as they were defined in part (a). The process of drawing a circle tangent to $OA$, $OB$, and the smallest circle is repeated forever. What fraction of the area of $\triangle AOB$ is covered by circles? Your answer should be in terms of $\theta$.

The result of part (c) can be applied to solve part (b). Can you see how?
Hint

(a) There are plenty of ways to approach this problem. One useful construction is to connect the centres of the circles to each other, then draw a perpendicular from each centre to the line \( OB \).

(b) It is possible to apply part (a) with \( \theta = 30^\circ \). While there are other ways to do this part, the easiest probably involves finding the sum of an infinite geometric series. You may want to look up how this is done.

(c) In some sense, this is a more general version of part (b). Finding the sum of a geometric series will be useful again here, but the terms of the series will be in terms of the variable \( \theta \). You might also find it useful to express the area of the triangle and the area of the largest circle in ways that are easy to compare.