Problem of the Month
Problem 7: April 2020

Problem
Define a function $f$ whose input and output are both lists of $n$ nonnegative integers by

$$f(a_1, a_2, \ldots, a_n) = (|a_1 - a_2|, |a_2 - a_3|, \ldots, |a_{n-1} - a_n|, |a_n - a_1|)$$

where, as usual, $|x|$ represents the absolute value of $x$.

For example, $f(1, 2, 3, 4) = (|1 - 2|, |2 - 3|, |3 - 4|, |4 - 1|) = (1, 1, 1, 3)$ and

$$f(2, 3, 5) = (|2 - 3|, |3 - 5|, |5 - 2|) = (1, 2, 3).$$

We will denote by $f^k$ the function that iterates the application of $f$ a total of $k$ times. For example,

$$f^4(1, 1, 1, 3) = f^3(0, 0, 2, 2) = f^2(0, 2, 0, 2) = f(2, 2, 2, 2) = (0, 0, 0, 0).$$

We will call the list $(a_1, a_2, \ldots, a_n)$ smooth if there is some $m$ for which $f^m(a_1, \ldots, a_n)$ is the list of $n$ zeros. That is, a list is smooth if some number of applications of $f$ will result in the list of all zeros. For example, $(1, 1, 1, 3)$ is smooth since $f^4(1, 1, 1, 3) = (0, 0, 0, 0)$, as demonstrated above.

(a) Find lists of length 5 and 7 that are not smooth.

(b) Show that for all odd integers $n \geq 1$ there exists a list $L$ of length $n$ that is not smooth.

(c) How many smooth lists $(a, b, c)$ are there with $a$, $b$, and $c$ each no larger than 100?

(d) Suppose $L$ is a list of length 4 consisting of only zeros and ones. Show that $L$ is smooth.

(e) Show that all lists of length 4 are smooth.
Hint

(a) Consider lists that only contain the integers 0 and 1.

(b) As with part (a), construct a list $L$ so that all of its entries are either 0 or 1. If the number of 1s in $L$ is a positive even number, what can you say about the number of 1s in $f(L)$?

(c) Supposing that $L$ is a list of three integers, how do the parities (parity refers to whether an integer is even or odd) of the integers in $L$ compare to the parities of integers in $f(L)$? You might want to consider what happens to lists $L$ with various combinations of even and odd integers. It may also be helpful to think about how things can be simplified if the integers $a$, $b$, and $c$ have a common factor.

(d) There are only 16 such lists, so you could show this by checking all of them.

(e) Compute $f^4(a, b, c, d)$ for a few lists $(a, b, c, d)$. What do you notice?