Problem 1: The triple (3, 4, 5) is a Pythagorean triple. Show that (5, 12, 13) and (7, 24, 25) are also Pythagorean triples.

Solution:
If $a = 5$, $b = 12$, and $c = 13$, then $a^2 + b^2 = 5^2 + 12^2 = 25 + 144 = 169$ and $c^2 = 13^2 = 169$. Therefore, $a^2 + b^2 = c^2$.

If $a = 7$, $b = 24$, and $c = 25$, then $a^2 + b^2 = 7^2 + 24^2 = 49 + 576 = 625$ and $c^2 = 25^2 = 625$. Therefore, $a^2 + b^2 = c^2$.

Problem 2:

(a) Build a Pythagorean triple that includes the odd integer 9 by following these steps:

(i) Determine $n$ such that $9^2 = 2n + 1$. (Answer: $n = \frac{81 - 1}{2} = 40$.)

(ii) Verify that $(n + 1)^2 - n^2 = 9^2$. (Answer: $41^2 - 40^2 = 1681 - 1600 = 81 = 9^2$.)

(iii) Write down a Pythagorean triple for which the smallest integer is 9. (Answer: Since $41^2 - 40^2 = 9^2$, we have $9^2 + 40^2 = 41^2$ and so (9, 40, 41) is a Pythagorean triple.)

(b) Build a Pythagorean triple that includes the odd integer 11 by following these steps:

(i) Determine $n$ such that $11^2 = 2n + 1$.

(ii) Verify that $(n + 1)^2 - n^2 = 11^2$.

(iii) Write down a Pythagorean triple for which the smallest integer is 11.

(c) Use the ideas from (a) and (b) to build Pythagorean triples that include the next four odd integers: 13, 15, 17, 19.

Solution:

(b) (i) We have $11^2 = 121 = 2n + 1$ exactly when $n = \frac{121 - 1}{2} = 60$.

(ii) When $n = 60$ we have $(n + 1)^2 - n^2 = 61^2 - 60^2 = 3721 - 3600 = 121 = 11^2$.

(iii) Since $61^2 - 60^2 = 11^2$, we have $11^2 + 60^2 = 61^2$ and so (11, 60, 61) is a Pythagorean triple involving the integer 11.

(c) For the integer 13: We have $13^2 = 169 = 2n + 1$ exactly when $n = \frac{169 - 1}{2} = 84$. We can verify that $85^2 - 84^2 = 13^2$ which means $13^2 + 84^2 = 85^2$ and so (13, 84, 85) is a Pythagorean triple involving the integer 13.

Using this same method, we can also obtain the following Pythagorean triples:

(15, 112, 113), (17, 144, 145), (19, 180, 181)
Problem 3:

(a) Consider the Pythagorean triple $\left(3, 4, 5\right)$. Show that if you multiply each integer in the triple by 2, then you obtain another Pythagorean triple.

 Solution:

If we multiply each integer in the triple $\left(3, 4, 5\right)$ by 2, then we obtain the triple $\left(6, 8, 10\right)$. We can check that this triple is a Pythagorean triple as follows: $6^2 + 8^2 = 36 + 64 = 100 = 10^2$.

(b) Use the idea from (a) to build another Pythagorean triple that includes the odd integer 9.

 Solution:

If we multiply each integer in the triple $\left(3, 4, 5\right)$ by 3, then we obtain the triple $\left(9, 12, 15\right)$. We can check that this triple is a Pythagorean triple as follows: $9^2 + 12^2 = 81 + 144 = 225 = 15^2$.

Note that we have now found two different Pythagorean triples that involve the odd number 9: $\left(9, 40, 41\right)$ and $\left(9, 12, 15\right)$.

(c) Show that for every positive integer $n$, the triple $\left(3n, 4n, 5n\right)$ is a Pythagorean triple.

 It is also true that $\left(5n, 12n, 13n\right)$ and $\left(7n, 24n, 25n\right)$ are Pythagorean triples.

 Solution:

First we note that for every positive $n$, the numbers $3n$, $4n$, and $5n$ are positive integers. Also, we have $(3n)^2 + (4n)^2 = 9n^2 + 16n^2 = 25n^2 = (5n)^2$. This means that the triple $\left(3n, 4n, 5n\right)$ is a Pythagorean triple.

Note: In a similar way, we can show that $(5n)^2 + (12n)^2 = 25n^2 + 144n^2 = 169n^2 = (13n)^2$ and $(7n)^2 + (24n)^2 = 49n^2 + 576n^2 = 625n^2 = (25n)^2$.

(d) Use the ideas from Problem 2 and Problem 3 to show that every integer from 4 to 20 is part of at least one Pythagorean triple.

 Solution:

We provide at least one triple for each integer. Some of the triples given below have already been justified earlier. See if you can determine how the other triples were built using the ideas from Problem 2 and Problem 3. For example, the second triple given for 10 was obtained by multiplying each integer in the Pythagorean triple $\left(5, 12, 13\right)$ by 2 and using the idea from Problem 3(c).

 4: $\left(3, 4, 5\right)$

 5: $\left(3, 4, 5\right)$, $\left(5, 12, 13\right)$

 6: $\left(6, 8, 10\right)$

 7: $\left(7, 24, 25\right)$

 8: $\left(6, 8, 10\right)$

 9: $\left(9, 40, 41\right)$, $\left(9, 12, 15\right)$

 10: $\left(6, 8, 10\right)$, $\left(10, 24, 26\right)$

 11: $\left(11, 60, 61\right)$

 12: $\left(5, 12, 13\right)$, $\left(9, 12, 15\right)$, $\left(12, 16, 20\right)$

 13: $\left(13, 84, 85\right)$

 14: $\left(14, 48, 50\right)$

 15: $\left(9, 12, 15\right)$, $\left(15, 112, 113\right)$

 16: $\left(12, 16, 20\right)$

 17: $\left(17, 144, 145\right)$

 18: $\left(18, 24, 30\right)$, $\left(18, 80, 82\right)$

 19: $\left(19, 180, 181\right)$

 20: $\left(12, 16, 20\right)$, $\left(20, 48, 52\right)$
Challenge Problem: Think about how you might use some of these ideas to show that every integer that is at least 3 is part of a Pythagorean triple. One possible approach is outlined below, but there are others:

(a) Odd numbers:

(i) Show that for every positive integer $n$, we have $(n + 1)^2 - n^2 = 2n + 1$.

![Image](image.png)

(ii) Use the identity from part (i) to explain why every odd integer that is at least 3 is part of a Pythagorean triple.

Solution:

(i) Method 1: Using the image provided, we see that the area of the largest square is represented by the quantity $(n + 1)(n + 1)$, the area of the medium square is represented by the quantity $(n)(n)$, the area of each of the two rectangles is represented by the quantity $(1)(n)$, and the area of the smallest square is represented by the quantity $(1)(1)$. Since the medium square, small square and two rectangular regions are used to form the larger square we must have

$$(n + 1)(n + 1) = (n)(n) + (1)(n) + (1)(1)$$

This simplifies to $(n+1)^2 = n^2+2n+1$ which can be rearranged to give $(n+1)^2−n^2 = 2n+1$.

Method 2: Using the distributive property, we have $(n+1)(n+1) = (n+1)(n) + (n+1)(1)$ which means

$$(n + 1)^2 = (n + 1)(n + 1) = (n + 1)(n) + (n + 1)(1) = n^2 + n + n + 1 = n^2 + 2n + 1$$

It follows that $(n + 1)^2 − n^2 = (n^2 + 2n + 1) − n^2 = 2n + 1$.

Method 3: If you have seen the formula for the difference of squares before, then you may see that

$$(n + 1)^2 − n^2 = ((n + 1) + n)((n + 1) − n) = (2n + 1)(1) = 2n + 1$$

(ii) We follow the method from Problem 2 for a general odd integer $k$ that is greater than 1: Let $n = \frac{k^2−1}{2}$. Since $k^2$ must also be an odd integer that is greater than 1, $k^2 − 1$ must be an even integer that is greater than 0. This means $n$ is a positive integer. Rearranging the equation gives $k^2 = 2n + 1$. Using the formula from (i), we get that

$$(n + 1)^2 − n^2 = 2n + 1 = k^2$$

for this value of $n$. Rearranging the equation above gives

$$k^2 + n^2 = (n + 1)^2$$

which shows that $(k, n, n+1)$ is a Pythagorean triple that includes the given odd integer $k$. 

(b) Even numbers:

(i) Show that for every positive integer \( n \), we have \((n + 2)^2 - n^2 = 4n + 4\).

(ii) Use the identity from part (i) to explain why every even integer that is at least 4 is part of a Pythagorean triple.

**Solution:**

(i) Since \((n + 2)^2 = (n + 2)(n + 2) = (n + 2)(n) + (n + 2)(2) = n^2 + 2n + 2n + 4 = n^2 + 4n + 4\) we have \((n + 2)^2 - n^2 = (n^2 + 4n + 4) - n^2 = 4n + 4\).

(ii) To show this you can follow the method from part (a) of the challenge problem. We do not give a full solution here, but instead outline the steps using an example. We can build a Pythagorean triple that includes the even integer 6 by following these steps:

- Determine \( n \) such that \( 6^2 = 4n + 4 \):
  Solving we get \( n = \frac{36 - 4}{4} = 8 \).
- From part (i) above, we know that for this \( n \) we will have \((n + 2)^2 - n^2 = 6^2\).
  We can verify this directly: \((8 + 2)^2 - 8^2 = 10^2 - 8^2 = 100 - 64 = 36 = 6^2\).
- This works shows that \((6, 8, 10)\) is a Pythagorean triple.

Suppose that \( k \) is an even integer that is greater than 2. If you can find a positive integer \( n \) such that \( k^2 = 4n + 4 \), then the steps above show that \((k, n, n + 2)\) is a Pythagorean triple. Can you see why there will always be such a value of \( n \)?

If \( k > 2 \) and is even then \( k^2 > 4 \) and is a multiple of 4, and so \( k^2 - 4 > 0 \) and is a multiple of 4. It follows that \( n = \frac{k^2 - 4}{4} \) is a positive integer and satisfies \( k^2 = 4k + 4 \) as needed!

*Parts (a) and (b) of the challenge problem show that every integer that is at least 3 is part of a Pythagorean triple. Can you explain why the integers 1 and 2 cannot be part of Pythagorean triples?*