Pythagorean Triples

The Pythagorean Theorem: In a right-angled triangle, where \( c \) represents the length of the hypotenuse and \( a \) and \( b \) represent the lengths of the two legs (the two shorter sides), the following equation is true:

\[
a^2 + b^2 = c^2
\]

Note: It is also true that any triangle with side lengths \( a \), \( b \), and \( c \) that satisfy the equation \( a^2 + b^2 = c^2 \) must be a right-angled triangle.

A Pythagorean triple is a triple of integers \((a, b, c)\) that satisfies \( a^2 + b^2 = c^2 \).

If a triangle is formed with integer side lengths \( a \), \( b \), and \( c \), then this triangle is a right-angled triangle exactly when \((a, b, c)\) is a Pythagorean triple.

**Problem 1:** The triple \((3, 4, 5)\) is a Pythagorean triple. If \( a = 3 \), \( b = 4 \), and \( c = 5 \), then

\[
a^2 + b^2 = 3^2 + 4^2 = 9 + 16 = 25 \quad \text{and} \quad c^2 = 5^2 = 25
\]

and therefore, \( a^2 + b^2 = c^2 \).

Show that \((5, 12, 13)\) and \((7, 24, 25)\) are also Pythagorean triples.

Is every positive integer a part of at least one Pythagorean triple? It turns out that the answer to this question is no. However, every positive integer that is at least 3 is a part of a Pythagorean triple. Let’s explore this idea.

**Problem 2:** Consider the Pythagorean triples from Problem 1: \((3, 4, 5)\), \((5, 12, 13)\), and \((7, 24, 25)\). Notice that the integers in the leftmost coordinates of the triples are the odd integers 3, 5, and 7. Do you notice a pattern in the other two integers in each triple? The remaining two integers in each triple are consecutive integers: 4 and 5, 12 and 13, 24 and 25. Let’s explore this pattern.

(a) Build a Pythagorean triple that includes the odd integer 9 by following these steps:

(i) Determine \( n \) such that \( 9^2 = 2n + 1 \). (Answer: \( n = \frac{81 - 1}{2} = 40 \).)

(ii) Verify that \((n + 1)^2 - n^2 = 9^2\). (Answer: \(41^2 - 40^2 = 1681 - 1600 = 81 = 9^2\).)

(iii) Write down a Pythagorean triple for which the smallest integer is 9. (Answer: Since \(41^2 - 40^2 = 9^2\), we have \(9^2 + 40^2 = 41^2\) and so \((9, 40, 41)\) is a Pythagorean triple.)
(b) Build a Pythagorean triple that includes the odd integer 11 by following these steps:

(i) Determine \( n \) such that \( 11^2 = 2n + 1 \).
(ii) Verify that \( (n + 1)^2 - n^2 = 11^2 \).
(iii) Write down a Pythagorean triple for which the smallest integer is 11.

(c) Use the ideas from (a) and (b) to build Pythagorean triples that include the next four odd integers: 13, 15, 17, 19.

Can you see how to do this for any odd integer? We explore this in the challenge problem.

Problem 3:

(a) Consider the Pythagorean triple \((3, 4, 5)\). Show that if you multiply each integer in the triple by 2, then you obtain another Pythagorean triple.

(b) Use the idea from (a) to build another Pythagorean triple that includes the odd integer 9.

(c) Show that for every positive integer \( n \), the triple \((3n, 4n, 5n)\) is a Pythagorean triple.

It is also true that \((5n, 12n, 13n)\) and \((7n, 24n, 25n)\) are Pythagorean triples.

(d) Use the ideas from Problem 2 and Problem 3 to show that every integer from 4 to 20 is part of at least one Pythagorean triple.

Challenge Problem: Think about how you might use some of these ideas to show that every integer that is at least 3 is part of a Pythagorean triple. One possible approach is outlined below, but there are others:

(a) Odd numbers:

(i) Show that for every positive integer \( n \), we have \( (n + 1)^2 - n^2 = 2n + 1 \).

(ii) Use the identity from part (i) to explain why every odd integer that is at least 3 is part of a Pythagorean triple.

(b) Even numbers:

(i) Show that for every positive integer \( n \), we have \( (n + 2)^2 - n^2 = 4n + 4 \).

(ii) Use the identity from part (i) to explain why every even integer that is at least 4 is part of a Pythagorean triple.

More Info:
Check out the CEMC at Home webpage on Tuesday, June 16 for a solution to Pythagorean Triples.