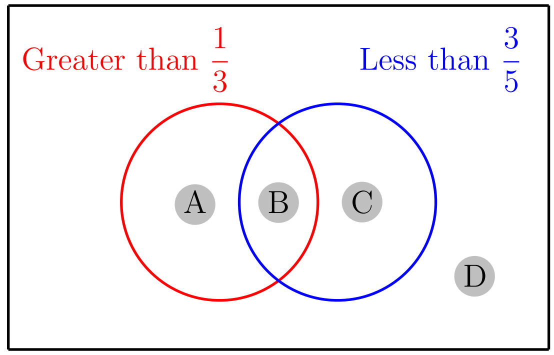
CEMC at Home

Grade 7/8 - Tuesday, May 26, 2020
Going in Circles - Solution

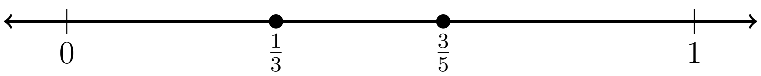
Problem 1

This Venn diagram has four regions. Place a fraction in as many of the regions as you can. Is it possible to find a fraction for each region?



Solution:

We have marked the four regions A, B, C, and D.
We plot the fractions on a number line as a reference:

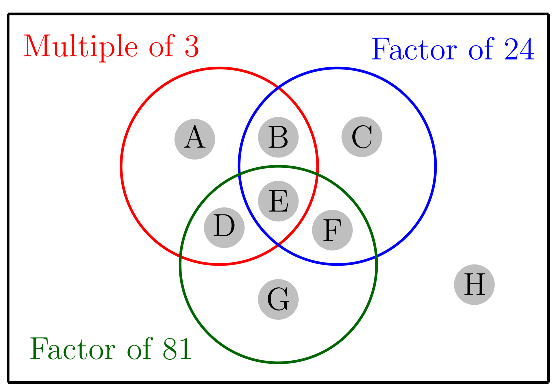


- Any fraction in Region A must be greater than $\frac{1}{3}$ and *not* less than $\frac{3}{5}$.
  This means the fraction must be greater than $\frac{1}{3}$ and greater than or equal to $\frac{3}{5}$.
  Some examples are $\frac{4}{5}$, $\frac{2}{3}$, and $\frac{3}{4}$. (Any fraction greater than or equal to $\frac{3}{5}$ works.)
- Any fraction in Region B must be greater than $\frac{1}{3}$ and less than $\frac{3}{5}$.
  Some examples are $\frac{1}{2}$, $\frac{2}{5}$, and $\frac{4}{7}$. (Any fraction between $\frac{1}{3}$ and $\frac{3}{5}$ works.)
- Any fraction in Region C must be less than $\frac{3}{5}$ and *not* greater than $\frac{1}{3}$.
  This means the fraction must be less than $\frac{3}{5}$ and less than or equal to $\frac{1}{3}$.
  Some examples are $\frac{1}{4}$, $\frac{1}{5}$, and $\frac{1}{6}$. (Any fraction less than or equal to $\frac{1}{3}$ works.)
- Any fraction in Region D must *not* be greater than $\frac{1}{3}$ and *not* be less than $\frac{3}{5}$.
  This means the fraction must be less than or equal to $\frac{1}{3}$ and greater than or equal to $\frac{3}{5}$.
  It is not possible to find such a fraction and so this region must remain empty.

Therefore, we can place a fraction in three of the four regions.
For example, we could place $\frac{4}{5}$ in region A, $\frac{2}{5}$ in region B, $\frac{1}{5}$ in region C, and no fraction in region D.

Problem 2

This Venn diagram has eight regions (seven regions "inside" at least one of the circles and one region "outside" all three circles). Place a positive integer in as many of the regions as you can. Is it possible to find a positive integer for each region?



Solution:

It is helpful if we first write out the factors of 24 and 81.

Factors of 24: 1, 2, 3, 4, 6, 8, 12, 24
Factors of 81: 1, 3, 9, 27, 81

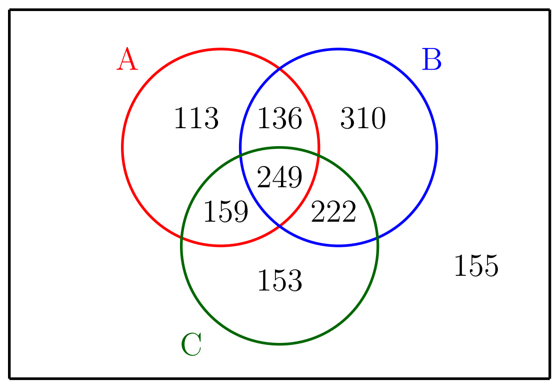
We have marked the eight regions A, B, C, D, E, F, G, and H. We can place a positive integer in each region except for region G.

- Any integer in Region A must be a multiple of 3 but not a factor of 24 and not a factor of 81. Some examples are 18, 21, and 30.
- Any integer in Region B must be a multiple of 3 and a factor of 24 but not a factor of 81. The only options are 6, 12, and 24.
- Any integer in Region C must be a factor of 24 but not a factor of 81 and not a multiple of 3. The only options are 2, 4, and 8.
- Any integer in Region D must be a multiple of 3 and a factor of 81 but not a factor of 24. The only options are 9, 27, and 81.
- Any integer in Region E must be a multiple of 3 and a factor of both 24 and 81. The only option is 3.
- Any integer in Region F must be a factor of both 24 and 81 but not a multiple of 3. The only option is 1.
- Any integer in Region G must be a factor of 81 but not a factor of 24 and not a multiple of 3. It is not possible to find such an integer, so this region must remain empty.
- Any integer in Region H must not be multiple of 3, not be a factor of 24, and not be a factor of 81. Some examples are 5, 7, and 10.

Problem 3

This Venn diagram has eight regions. Place a positive three-digit integer in as many of the regions as you can. Is it possible to find a three-digit integer for each region?

A: 5 is a factor of the sum of the digits
B: The product of the digits is even
C: The mean of the digits is an integer



Solution:

Positive three-digit integers have been placed in the regions in the diagram above. It is possible to place a number in each of the eight regions, and there are other choices you could have made.

There are many ways to go about finding these numbers. You can choose three-digit numbers randomly and then test them to see in which region they belong, hoping to eventually find one for every region, or you can try to reason what digits in each of the regions must look like.

For example, you can note that the product of the digits of an integer is even exactly when the number has at least one even digit. Because of this we know that any number placed within the circle marked B must have at least one even digit, and every number placed outside of this circle must have three odd digits.

We can further note that three odd digits must have an odd sum. So if a number is *outside* circle B but *inside* circle A, then it must have three odd digits that add to an odd multiple of 5. (In fact, they must add to either 5 or 15. Can you see why?) The numbers 113 and 159 both have this property and exactly one of them has the additional property that the mean of its digits is equal to an integer. (The mean of the digits of 113 is $\frac{1+1+3}{3} = \frac{5}{3}$, which is not an integer, and the mean of the digits of 159 is $\frac{1+5+9}{3} = \frac{15}{3} = 5$, which is an integer.)

A combination of reasoning and some trial and error is a good approach for this problem!