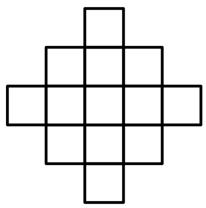
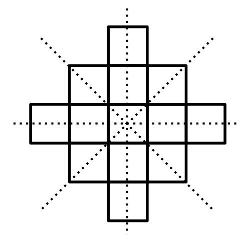
CEMC at Home

Grade 7/8 - Friday, May 22, 2020
Mirror, Mirror - Solution

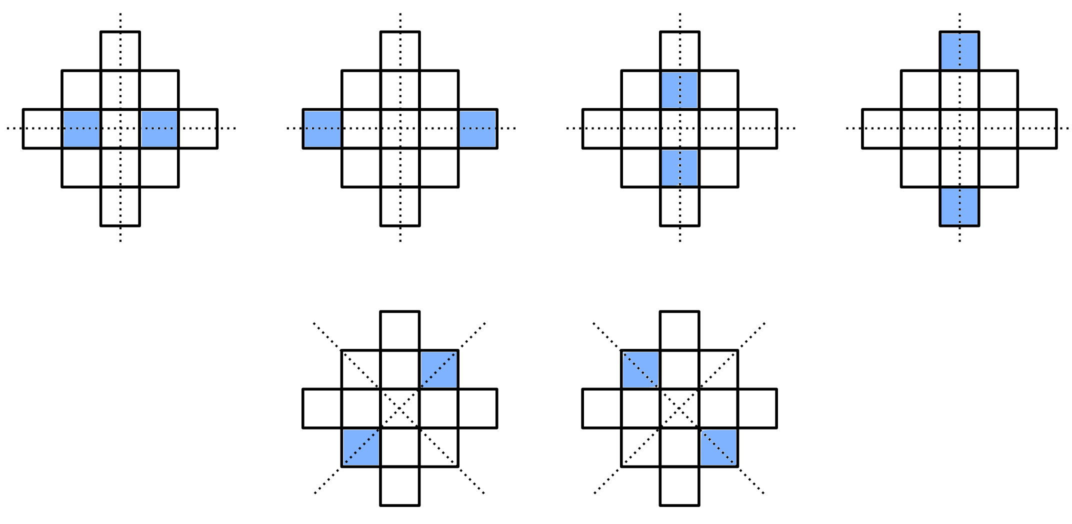
Thirteen identical squares are arranged as shown in the figure below on the left. Notice that the design with no shaded squares has exactly four lines of symmetry, as shown in the image below on the right.





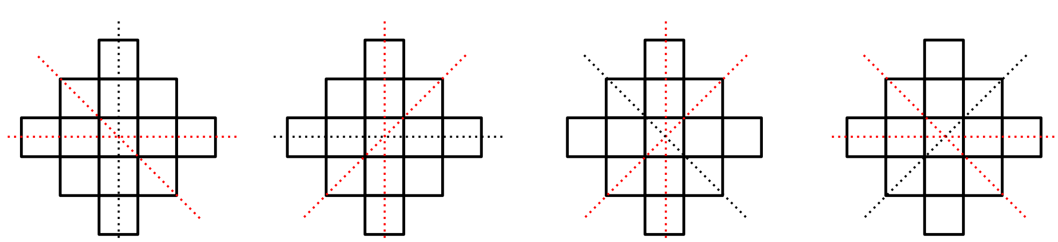
Problem 1: How many different designs are there with exactly two shaded squares that have exactly two lines of symmetry?

Solution: There are six different designs. They are shown below.



Problem 2: Is it possible to create a design that has exactly three lines of symmetry? If so, draw one. If not, explain why this is not possible.

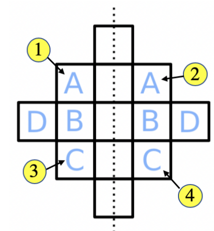
Solution: It is not possible. First, we notice that any group of three of the four lines of symmetry must include a diagonal line *and* a horizontal or vertical line. The four possible combinations of three lines of symmetry are shown below.



Group 1 Group 2 Group 3 Group 4

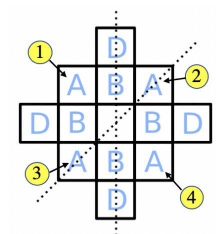
It turns out that any design that has a diagonal line of symmetry *and* a vertical or horizontal line of symmetry must actually have all four lines of symmetry. (This means it cannot be possible to create a design with exactly three lines of symmetry.)

Suppose we have a design that has the three lines of symmetry in Group 2. We will use letters to show the squares that must have the same shading, based on the lines of symmetry. For example, the squares marked with the letter "A" must be either all shaded, or all not shaded.



Since the design has the vertical line of symmetry, the design must have the symmetry indicated in the top image on the right.

Since the design *also* has the diagonal line of symmetry from the lower left corner to the upper right corner, the design must have the additional symmetry shown in the bottom image on the right.

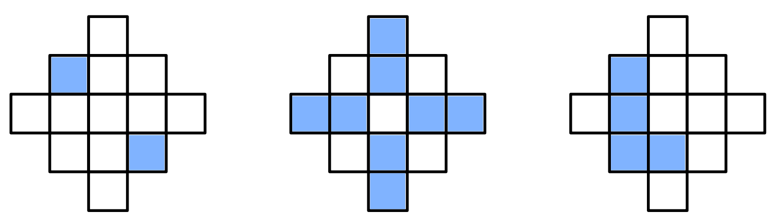


For example, the vertical line of symmetry tells us that the squares marked with 1 and 2 must look the same and that the squares marked with 3 and 4 must look the same. The diagonal line of symmetry tells us that the squares marked with 1 and 4 must look the same. Putting this together, we see that all four of these squares must look the same and so are marked with the same letter, A.

From this, we can see that the design actually has all four lines of symmetry.

This means any design with these two lines of symmetry will actually have all four lines of symmetry. This argument works for Group 2 and Group 3. The argument for Group 1 and Group 4 is similar.

Problem 3: A design has rotational symmetry if we can rotate it about its centre less than a full turn and produce a design that looks identical to the original design. The first two designs below have rotational symmetry but the last design does not.



How many different designs are there with exactly three shaded squares that have rotational symmetry and at least one line of symmetry?

Solution: There are six different designs. They are shown below.