Problem 1: Valerie has some cubes and some square-based pyramids.

Valerie put some number of each of these solids in a bag. In total, the solids in her bag have 68 square faces, 56 triangular faces, and no other types of faces. How many of each shape did Valerie put in her bag?

There are several ways to solve this problem. Three different solutions are shown below.

Method 1: Make a table

<table>
<thead>
<tr>
<th>Number of Cubes</th>
<th>Number of Square Faces</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
</tr>
<tr>
<td>3</td>
<td>18</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>9</td>
<td>54</td>
</tr>
<tr>
<td>10</td>
<td>60</td>
</tr>
<tr>
<td>11</td>
<td>66</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Number of Pyramids</th>
<th>Number of Square Faces</th>
<th>Number of Triangular Faces</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>12</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>14</td>
<td>14</td>
<td>56</td>
</tr>
</tbody>
</table>

Notice that if we have 9 cubes and 14 pyramids, then we have $54 + 14 = 68$ square faces and 56 triangular faces as needed. So Valerie could have 9 cubes and 14 pyramids in her bag.

Can you use the table to convince yourself that this is the only possibility for the combination of objects in Valerie’s bag? We will justify this in the following two solutions.

This method is time consuming because we needed to fill in many rows of the tables before we found a combination that worked. The other methods are more efficient for solving this problem.

Method 2: Use logic

Since cubes do not have triangular faces, the triangular faces must have all come from the pyramids. Each pyramid has 4 triangular faces. Since $56 \div 4 = 14$, there must be 14 pyramids in order to have 56 triangular faces in total.

Each pyramid has 1 square face, so the 14 pyramids have 14 square faces among them. There are 68 square faces in total. Since $68 - 14 = 54$, this means the remaining 54 square faces must come from cubes. Each cube has 6 square faces. Since $54 \div 6 = 9$, there must be 9 cubes in order to have 54 square faces among the cubes.

This means Valerie must have 9 cubes and 14 pyramids in her bag.
Method 3: Use variables

Let \( c \) represent the number of cubes and \( s \) represent the number of square-based pyramids in Valerie’s bag.

Since each cube has 0 triangular faces, and each pyramid has 4 triangular faces, the number of triangular faces in the bag must be \( 4 \times s \). Since there are 56 triangular faces in total, we must have

\[
4 \times s = 56
\]

Notice that \( 4 \times 14 = 56 \). The only positive integer that satisfies this equation is \( s = 14 \) and so we know the number of pyramids must be 14.

Since each cube has 6 square faces and each pyramid has 1 square face, the number of square faces in the bag must be \( 6 \times c + 1 \times s \). Since there are 68 square faces in total, we must have

\[
6 \times c + 1 \times s = 68
\]

Remember that \( s = 14 \) so substituting this value we get

\[
6 \times c + 1 \times 14 = 68
\]

From this we see that \( 6 \times c \) must equal 54 which means \( c \) must equal 9.

Therefore, Valerie must have 9 cubes and 14 pyramids in her bag.

**Problem 2:** Max has some square-based pyramids and some triangular prisms whose rectangular faces are actually squares.

Max put some number of each of these solids in a bag. In total, the solids in their bag have 13 square faces, some number of triangular faces, and no other types of faces. How many different combinations of solids could Max have put in their bag?

There are several ways to solve this problem. Two different solutions are shown below.

Method 1: Use diagrams

The squares below represent the 13 square faces in Max’s bag.

\[
\begin{array}{cccccccccccc}
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\end{array}
\]

We know each triangular prism has 3 square faces and each pyramid has 1 square face. So we want to figure out how many ways we can put the 13 squares into groups of 3 and 1. The diagram below shows all possible ways to do this, where each group of 3 is circled.

\[
\begin{array}{cccccccccccc}
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\end{array} \quad \rightarrow \quad 0 \text{ triangular prisms and 13 pyramids}
\]

\[
\begin{array}{cccccccccccc}
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\end{array} \quad \rightarrow \quad 1 \text{ triangular prism and 10 pyramids}
\]

\[
\begin{array}{cccccccccccc}
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\end{array} \quad \rightarrow \quad 2 \text{ triangular prisms and 7 pyramids}
\]

\[
\begin{array}{cccccccccccc}
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\end{array} \quad \rightarrow \quad 3 \text{ triangular prisms and 4 pyramids}
\]

\[
\begin{array}{cccccccccccc}
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\end{array} \quad \rightarrow \quad 4 \text{ triangular prisms and 1 pyramid}
\]

So there are 5 different possible combinations for the solids in Max’s bag.
Solution 2: Use variables

Let $t$ represent the number of triangular prisms and $s$ represent the number of square-based pyramids in Max’s bag. Each triangular prism has 3 square faces and each pyramid has 1 square face. Since there are 13 square faces in total, we can write the following equation.

$$3 \times t + 1 \times s = 13$$

Notice that there is only one equation and it has two variables. This means we cannot solve this equation in the same way we might solve other equations with only one variable. It also means that the equation may have more than one solution. To find some solutions, we start testing some values.

Suppose there are 0 triangular prisms (and so $t = 0$). Then the equation becomes

$$3 \times 0 + 1 \times s = 13 \quad \text{or} \quad 1 \times s = 13$$

This tells us that in this case we must have $s = 13$.

Suppose there is 1 triangular prism (and so $t = 1$). Then the equation becomes

$$3 \times 1 + 1 \times s = 13 \quad \text{or} \quad 3 + 1 \times s = 13$$

This tells us that in this case we must have $3 + s = 13$ which means $s = 10$.

What are the values of $s$ if there are 2, 3, or 4 triangular prisms (that is, if $t = 2$, $t = 3$, or $t = 4$)? Can there be 5 or more triangular prisms?

It turns out that there are five solutions, and they are shown below:

$$3 \times 0 + 1 \times 13 = 13 \rightarrow 0 \text{ triangular prisms and 13 pyramids}$$
$$3 \times 1 + 1 \times 10 = 13 \rightarrow 1 \text{ triangular prism and 10 pyramids}$$
$$3 \times 2 + 1 \times 7 = 13 \rightarrow 2 \text{ triangular prisms and 7 pyramids}$$
$$3 \times 3 + 1 \times 4 = 13 \rightarrow 3 \text{ triangular prisms and 4 pyramids}$$
$$3 \times 4 + 1 \times 1 = 13 \rightarrow 4 \text{ triangular prisms and 1 pyramid}$$

Notice that there cannot be 5 or more triangular prisms in the bag. 5 triangular prisms will contribute $3 \times 5 = 15$ square faces which is more than the total of 13.

So there are 5 different possible combinations for the solids in Max’s bag.

Did you use a different approach to solve Problem 2? Is there a way to use a table or other reasoning to solve this problem?

More Info:

Equations with more than one variable like the one in the solution to Problem 2 are called Diophantine equations. For more practice finding solutions to Diophantine equations, check out this lesson in the CEMC Courseware.