Each problem below contains a positive integer that has at least one digit missing. A square is used in place of each missing digit.

To help us solve these problems, we will use various divisibility tests as outlined in the problem statement:

- A whole number is divisible by 3 exactly when the sum of its digits is divisible by 3.
- A whole number is divisible by 4 exactly when its last two digits (tens and units, in order) form a two-digit number that is divisible by 4.
- A whole number is divisible by 6 exactly when it is divisible by both 2 and 3.
- A whole number is divisible by 12 exactly when it is divisible by both 3 and 4.

Problem 1: The two-digit number 5□ is divisible by 2. How many possibilities are there for this two-digit number?

Solution: Since the number 5□ is divisible by 2, its units (ones) digit must be 0, 2, 4, 6, or 8. So there are five possibilities for the given two-digit number. They are 50, 52, 54, 56, and 58.

Problem 2: The four-digit number 6□53 is divisible by 3. How many possibilities are there for this four-digit number?

Solution: There are 10 possibilities for the digit in the box, and these result in the following four-digit numbers:

6053, 6153, 6253, 6353, 6453, 6553, 6653, 6753, 6853, 6953

One quick way to check which of these numbers is divisible by 3 is to calculate the sum of their digits and check whether or not this sum is divisible by 3.

Since 6 + 0 + 5 + 3 = 14 and 14 is not divisible by 3, the number 6053 is not divisible by 3.

On the other hand, 6 + 1 + 5 + 3 = 15 and 15 is divisible by 3, and so the number 6153 is divisible by 3.

Checking the sum of the digits of the other eight numbers, we find that only the numbers 6153, 6453, and 6753 have the sum of their digits divisible by 3.

This means there are three possibilities for the given number. They are 6153, 6453, and 6753.

Problem 3: The four-digit number 4□3□ is divisible by 6. How many possibilities are there for this four-digit number? Note that the two missing digits don’t need to be the same.

Solution: Since the number 4□3□ is divisible by 6, it must be divisible by both 2 and 3. The fact that it is divisible by 2 tells us that its units digit must be 0, 2, 4, 6, or 8. Let’s look at each of these cases. See the next page for the case work.
• Case 1: The units digit is 0.
  In this case the number looks like 4□30. Since the number is also divisible by 3, we know that the sum of its digits must be divisible by 3.
  – If the missing digit is 2, then the sum of the digits is 9, which is divisible by 3.
  – If the missing digit is 5, then the sum of the digits is 12, which is divisible by 3.
  – If the missing digit is 8, then the sum of the digits is 15, which is divisible by 3.
  None of the other possible digits result a sum that is divisible by 3.

• Case 2: The units digit is 2.
  In this case the number looks like 4□32. Since the number is also divisible by 3, we know that the sum of its digits must be divisible by 3.
  – If the missing digit is 0, then the sum of the digits is 9, which is divisible by 3.
  – If the missing digit is 3, then the sum of the digits is 12, which is divisible by 3.
  – If the missing digit is 6, then the sum of the digits is 15, which is divisible by 3.
  – If the missing digit is 9, then the sum of the digits is 18, which is divisible by 3.
  None of the other possible digits result a sum that is divisible by 3.

• Case 3: The units digit is 4.
  In this case the number looks like 4□34. Since the number is also divisible by 3, we know that the sum of its digits must be divisible by 3.
  – If the missing digit is 1, then the sum of the digits is 12, which is divisible by 3.
  – If the missing digit is 4, then the sum of the digits is 15, which is divisible by 3.
  – If the missing digit is 7, then the sum of the digits is 18, which is divisible by 3.
  None of the other possible digits result a sum that is divisible by 3.

• Case 4: The units digit is 6.
  In this case the number looks like 4□36.
  Doing similar work as the other cases, we find that only the digits 2, 5, and 8 produce digit sums that are divisible by 3.

• Case 5: The units digit is 8.
  In this case the number looks like 4□38.
  Doing similar work as the other cases, we find that only the digits 0, 3, 6, 9 produce digit sums that are divisible by 3.

We count that there are $3 + 4 + 3 + 3 + 4 = 17$ possibilities for the four-digit number 4□3□.

If we would like, we can also list them:

4230, 4530, 4830, 4032, 4332, 4632, 4932, 4134, 4434, 4734, 4236, 4536, 4836, 4038, 4338, 4638, 4938
**Challenge Problem:** The five-digit number □5□□2 is less than 30,000 and is divisible by 12. How many possibilities are there for this five-digit number?

**Solution:** Since the number □5□□2 is less than 30,000, the first digit must be either 1 or 2.

Since the number □5□□2 is divisible by 12, it must be divisible by both 3 and 4.

Since it is divisible by 4, the last two digits must form a two-digit number that is divisible by 4. The two-digit numbers of the form □2 that are divisible by 4 are 12, 32, 52, 72, and 92.

So we know that the number must “start” in one of two ways: with the digits 15 or the digits 25.

We also know that the number must “end” in one of five ways: with the digits 12, 32, 52, 72, or 92.

For example, one possibility is that the number is of the following form:

15□12

There are many different numbers that satisfy these properties! Our goal is to find the ones that are also divisible by 3.

Notice that all of these integers must have one of the following 10 forms shown in the table below. (Can you see why?) Similar to the previous problem, we determine which values when substituted for the missing digit result in the sum of the five digits in the number being divisible by 3.

<table>
<thead>
<tr>
<th>Form of the number</th>
<th>Digits resulting in a number divisible by 3</th>
<th>Count</th>
</tr>
</thead>
<tbody>
<tr>
<td>15□12</td>
<td>0, 3, 6, 9</td>
<td>4</td>
</tr>
<tr>
<td>15□32</td>
<td>1, 4, 7</td>
<td>3</td>
</tr>
<tr>
<td>15□52</td>
<td>2, 5, 8</td>
<td>3</td>
</tr>
<tr>
<td>15□72</td>
<td>0, 3, 6, 9</td>
<td>4</td>
</tr>
<tr>
<td>15□92</td>
<td>1, 4, 7</td>
<td>3</td>
</tr>
<tr>
<td>25□12</td>
<td>2, 5, 8</td>
<td>3</td>
</tr>
<tr>
<td>25□32</td>
<td>0, 3, 6, 9</td>
<td>4</td>
</tr>
<tr>
<td>25□52</td>
<td>1, 4, 7</td>
<td>3</td>
</tr>
<tr>
<td>25□72</td>
<td>2, 5, 8</td>
<td>3</td>
</tr>
<tr>
<td>25□92</td>
<td>0, 3, 6, 9</td>
<td>4</td>
</tr>
</tbody>
</table>

Adding up how many numbers we get in each of the 10 cases above, we see that there are

\[4 \times 4 + 6 \times 3 = 34\]

possibilities for the given integer.

*This is probably more numbers than you would want to write out!*

**Note:** We solved this challenge problem which involves checking whether five-digit numbers are divisible by 12 without actually attempting to divide a single five-digit number by 12. This shows the power of the divisibility tests! If you work through all of the details it takes to complete the table given above, then you will see that these tests allow us to replace questions like

“Is 25992 divisible by 3?”

with questions like

“Is the sum 2 + 5 + 9 + 9 + 2 = 27 divisible by 3?”