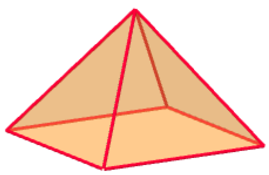
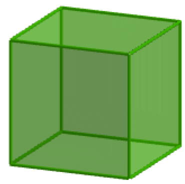
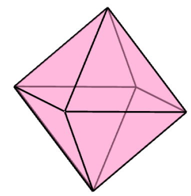
CEMC at Home

Grade 11/12 - Friday, May 22, 2020
Euler Characteristic - Part 1

A *polyhedron* (plural: polyhedra or polyhedrons) is a three-dimensional solid with polygons for its faces. Below are three examples of polyhedra: a square-based pyramid, a cube, and an octahedron.

A polyhedron has vertices, edges, and faces. Given a polyhedron, we let V, E, and F denote the number of vertices, edges, and faces, respectively, of the polyhedron.

In this activity, we will explore the relationship between the values of V, E, and F.

Example

Verify that the square-based pyramid has 5 vertices, 8 edges, and 5 faces. If we calculate the value of $V - E + F$ then we get $5 - 8 + 5 = 2$.

Example

Verify that the cube has 8 vertices, 12 edges, and 6 faces. If we calculate the value of $V - E + F$ then we get $8 - 12 + 6 = 2$.

Question

What is the value of $V - E + F$ for the octahedron?
You should get an answer of 2, again! Confirm this for yourself.

It seems unlikely that it is just a coincidence that all three of these polyhedra produce the same value of $V - E + F$. Is there some reason to believe that this will always happen?

> The **Euler characteristic** of a polyhedron, denoted χ, is defined to be the value of $V - E + F$. It turns out that $\chi = 2$ for every (convex) polyhedron.

Explaining why the Euler characteristic is always 2 is challenging, and we will explore this idea a bit further in next week's activity. You can use this fact, when needed, to solve the following problems.

Problem 1

Verify directly that $\chi = 2$ for a tetrahedron, a dodecahedron, and an icosahedron.
You may need to look up one or two of these platonic solids first!

Problem 2

A particular polyhedron has 26 faces and has twice as many edges as vertices. How many edges must the polyhedron have?

Problem 3

An Elongated Pentagonal Orthocupolarotunda is a polyhedron with exactly 37 faces, 15 of which are squares, 7 of which are regular pentagons, and 15 of which are triangles. How many vertices does it have?

Problem 4

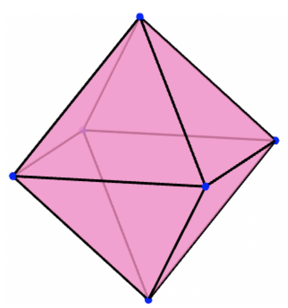
A polyhedron is formed with exactly P pentagons, exactly H hexagons, and no other polygons as its faces and has the property that three polygonal faces meet at each vertex.

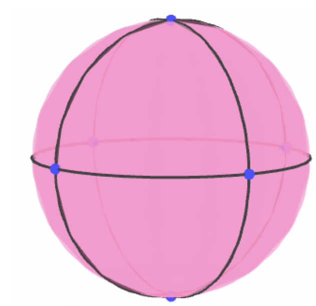
1. Explain why it is true that $E = \dfrac{5P + 6H}{2}$.
2. Explain why it is true that $V = \dfrac{5P + 6H}{3}$.
3. Using the fact that $\chi = 2$ for this polyhedron, show that it must be the case that $P = 12$.

A standard soccer ball can be thought of as an "inflation" of such a polyhedron.

Further Discussion

The idea of "inflating" a polyhedron can be used to help us understand why the Euler characteristic of a convex polyhedron is always 2. Imagine "inflating" a polyhedron, as if its surface is elastic like a balloon. For example, if we "inflate" the octahedron, then we obtain a sphere as shown below. We visualize the vertices, edges, and faces of the octahedron on the spherical surface.





Can you visualize what would happen if we "inflated" the pyramid or the cube in a similar way?

It can be helpful to have a common way in which to view all possible polyhedra; we can identify each polyhedron with how it looks after it is "inflated" to form a sphere. Notice that the resulting figure looks like a "tiling" of a spherical surface that uses polygon-like shapes as the tiles. (Of course, these shapes are not flat as they tile a curved surface.) For this reason, we call the result of "inflating" a polyhedron a *polygonization of a sphere*. This type of model gives us a nice way to compare polyhedra and their values of χ. We will revisit this idea in next Friday's activity.

Extra Problems to Think About

- Choose one face in the polygonization of a sphere shown above. Add two new vertices on two different edges of this face and join the new vertices with a new edge. How does this alteration affect the values of V, E, and F? What is the value of χ for this new polygonization?
- Choose one vertex in the polygonization of a sphere shown above. Remove this vertex and all of the edges that are directly connected to it. How does this alteration affect the values of V, E, and F? What is the value of χ for this new polygonization? (What is the polyhedron corresponding to this new polygonization? Think about "deflating the sphere".)

More Info: Check out the CEMC at Home webpage on Friday, May 29 for solutions to the problems in this activity and further discussion of polygonizations of surfaces and the Euler characteristic.