When solving problems we may encounter a polynomial with integer coefficients that needs to be factored. You may have learned some techniques for factoring polynomials that use long division of polynomials. In this activity we will factor some polynomials without using long division.

**Definition:** Suppose we have a polynomial in the variable $x$. If the polynomial evaluates to 0 when $x = a$, then we say that $a$ is a root of the polynomial.

**The Factor Theorem:** If $a$ is a root of a polynomial, then $x - a$ is a factor of the polynomial.

**Example 1**
The number $x = 3$ is a root of the polynomial $x^2 - x - 6$ since $3^2 - 3 - 6 = 0$. The factor theorem tells us that the polynomial $x - 3$ is a factor of the polynomial $x^2 - x - 6$. We can check that indeed $x^2 - x - 6 = (x - 3)(x + 2)$.

**Example 2**
The number $x = -1$ is a root of the polynomial $x^3 + 5x^2 + 8x + 4$ since $(-1)^3 + 5(-1)^2 + 8(-1) + 4 = 0$. The factor theorem tells us that the polynomial $x - (-1) = x + 1$ is a factor of the polynomial $x^3 + 5x^2 + 8x + 4$. We can check that indeed $x^3 + 5x^2 + 8x + 4 = (x + 1)(x^2 + 4x + 4)$.

*How might we find this other quadratic factor?*

In this activity, we will focus on factoring polynomials for which all but possibly two of the roots of the polynomial are integers; however, the techniques for factoring that we present below can also be useful in other situations.

**Factoring Method**

How do we go about factoring a polynomial with integer coefficients?

Let’s say we are factoring the cubic polynomial $2x^3 - x^2 - 7x + 6$. If $x = a$ is a root of this polynomial, then $x - a$ is a factor of the polynomial. If $a$ is an integer, then when we factor out $x - a$, we are left with some quadratic polynomial $Ax^2 + Bx + C$ with $A$, $B$, and $C$ integers as shown

$$2x^3 - x^2 - 7x + 6 = (x - a)(Ax^2 + Bx + C)$$

If we expand the product on the right side and compare its terms to the like terms on the left side, we observe the following:

- The only term on the right side without an $x$ in it will be the term $-aC$. This means $6 = -aC$. Since $a$ and $C$ are both integers, $a$ must be a factor of 6.
- The only term on the right side with a power of $x^3$ comes from multiplying the term $x$ by the term $Ax^2$. This means the term $2x^3$ must be equal to the term $Ax^3$ and so $A = 2$.
- There are two terms on the right side with a power of $x^2$, and they come from multiplying the term $x$ by the term $Bx$ and the term $-a$ by the term $Ax^2$. This means the term $-x^2$ on the left must be equal to $(x)(Bx) + (-a)(Ax^2)$ or $Bx^2 - aAx^2$. 
Using these three observations, we can factor the polynomial completely! Start by testing all of the factors of 6 to find an integer root $x = a$, and then use this value of $a$ along with the other two observations to solve for the coefficients $A$, $B$, and $C$. The full process is outlined in the examples below.

**Example 3:** Factor the cubic polynomial $2x^3 - x^2 - 7x + 6$.

The factors of 6 are $±1, ±2, ±3$ and $±6$. Using these factors, we determine that 1 is a root of the polynomial and so $x - 1$ is a factor. When we factor out $x - 1$ we will be left with a quadratic which we will call $Ax^2 + Bx + C$.

The $2x^3$ term from our original polynomial comes from multiplying $x$ by $Ax^2$.
Since $2x^3$ equals $Ax^3$ we must have $A = 2$. The constant term 6 from our original polynomial comes from multiplying $-1$ by $C$ and so $C = -6$.
The $-x^2$ term from our original polynomial comes from multiplying $x$ by $Bx$ and adding it to $-1$ times $2x^2$. Since $-x^2$ equals $Bx^2 - 2x^2$, we must have $B = 1$.
**Note that we didn’t use the $-7x$ term from our original polynomial, but it can be used to check that we didn’t make a mistake.**

Finally, we factor the resulting quadratic using standard factoring techniques.

<table>
<thead>
<tr>
<th>Example 3: Factor the cubic polynomial $2x^3 - x^2 - 7x + 6. $</th>
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<tbody>
<tr>
<td>$2x^3 - x^2 - 7x + 6 = (x - 1)(Ax^2 + Bx + C)$</td>
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<tr>
<td>$= (x - 1)(2x^2 + Bx - 6)$</td>
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<tr>
<td>$= (x - 1)(2x^2 + x - 6)$</td>
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<tr>
<td>$= (x - 1)(x + 2)(2x - 3)$</td>
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**Example 4:** Factor the quartic polynomial $6x^4 - 7x^3 - 13x^2 + 4x + 4$.

The factors of 4 are $±1, ±2$ and $±4$. Using these factors we determine that 2 is a root of our polynomial and so $x - 2$ is a factor.

We use the $6x^4$ term from our original polynomial to determine that $A = 6$ and the constant term 4 from our original polynomial to determine that $D = -2$. **Notice that $6x^4$ equals $(x)(Ax^3)$ and 4 equals $(-2)(D)$.**

We use the $-7x^3$ term from our original polynomial to determine that $B = 5$. **Notice that $-7x^3$ equals $(-2)(6x^3) + (x)(Bx^2)$.**

We use the $4x$ term from our original polynomial to determine that $C = -3$. **Notice that $4x$ equals $(x)(-2) + (2)(Cx)$.**

For the rest of our solution we ignore the $(x - 2)$ factor and focus on factoring the cubic $6x^3 + 5x^2 - 3x - 2$. **Remember that we have already discussed how to factor a cubic.** The factors of $-2$ are $±1$ and $±2$. Using these factors we determine that $-1$ is a root of this cubic and so $x + 1$ is a factor, and then we proceed as in Example 3.

<table>
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<th>Example 4: Factor the quartic polynomial $6x^4 - 7x^3 - 13x^2 + 4x + 4. $</th>
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<tr>
<td>$6x^4 - 7x^3 - 13x^2 + 4x + 4 = (x - 2)(Ax^3 + Bx^2 + Cx + D)$</td>
</tr>
<tr>
<td>$= (x - 2)(6x^3 + Bx^2 + Cx - 2)$</td>
</tr>
<tr>
<td>$= (x - 2)(6x^3 + 5x^2 + Cx - 2)$</td>
</tr>
<tr>
<td>$= (x - 2)(6x^3 + 5x^2 - 3x - 2)$</td>
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<tr>
<td>$= (x - 2)(x + 1)(Ex^2 + Fx + G)$</td>
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Use these ideas to solve the following problems.

1. Factor $x^3 + 7x^2 + 11x + 5$. 
2. Factor $x^4 + 5x^3 - 3x^2 - 17x - 10$. **Hint. Start by verifying that $x = 2$ is a root.** 
3. Factor $4x^4 - 16x^3 + x^2 + 39x - 18$. 
4. Given that $(Ax^2 + Bx + C)(3x^2 + Dx - 2) = 6x^4 + 3x^3 - 40x^2 + 2x + 4$, determine the values of $A, B, C$ and $D$. 

More Info:

Check out the CEMC at Home webpage on Tuesday, May 12 for a solution to Factoring Polynomials without Division.

When finding the roots of these polynomials we looked at a special case of the Rational Roots Theorem. To learn more about the Rational Roots Theorem check out the lesson Factoring Polynomials Using the Factor Theorem from the CEMC Advanced Functions and Pre-Calculus courseware.