CEMC at Home

Grade 9/10 - Monday, June 8, 2020
Contest Day 6

Today's resource features one question from the recently released 2020 CEMC Mathematics Contests.

2020 Fryer Contest, #3

In a *Dlin sequence*, the first term is a positive integer and each term after the first is calculated by adding 1 to the previous term in the sequence, then doubling the result. For example, the first seven terms of the Dlin sequence with first term 4 are:

$$4, 10, 22, 46, 94, 190, 382$$

(a) The 5^{th} term in a Dlin sequence is 142. What are the 4^{th} and 6^{th} terms in the sequence?

(b) Determine all possible first terms which give a Dlin sequence that includes 1406.

(c) Which possible first terms from 10 to 19 inclusive produce a Dlin sequence in which all terms after the first have the same ones (units) digit?

(d) Determine the number of positive integers between 1 and 2020, inclusive, that can be the third term in a Dlin sequence.

More Info:

Check out the CEMC at Home webpage on Monday, June 15 for a solution to the Contest Day 6 problem.

CEMC at Home

Grade 9/10 - Monday, June 8, 2020
Contest Day 6 - Solution

A solution to the contest problem is provided below.

2020 Fryer Contest, #3

In a *Dlin sequence*, the first term is a positive integer and each term after the first is calculated by adding 1 to the previous term in the sequence, then doubling the result. For example, the first seven terms of the Dlin sequence with first term 4 are:

$$4, 10, 22, 46, 94, 190, 382$$

(a) The 5^{th} term in a Dlin sequence is 142. What are the 4^{th} and 6^{th} terms in the sequence?

(b) Determine all possible first terms which give a Dlin sequence that includes 1406.

(c) Which possible first terms from 10 to 19 inclusive produce a Dlin sequence in which all terms after the first have the same ones (units) digit?

(d) Determine the number of positive integers between 1 and 2020, inclusive, that can be the third term in a Dlin sequence.

Solution:

(a) If the 5^{th} term in a Dlin sequence is 142, then the 6^{th} term is $(142 + 1) \times 2 = 143 \times 2 = 286$.
To determine the 4^{th} term in the sequence given the 5^{th}, we "undo" adding 1 followed by doubling the result by first dividing the 5^{th} term by 2 and then subtracting 1 from the result.
To see this, consider that if two consecutive terms in a Dlin sequence are a followed by b, then $b = (a + 1) \times 2$.
To determine the operations needed to find a given b (that is, to move backward in the sequence), we rearrange this equation to solve for a.

$$\begin{aligned} b &= (a+1) \times 2 \\ \tfrac{b}{2} &= a+1 \\ \tfrac{b}{2} - 1 &= a \end{aligned}$$

Thus if the 5^{th} term in the sequence is 142, then the 4^{th} term is $\frac{142}{2} - 1 = 71 - 1 = 70$.
(We may check that the term following 70 is indeed $(70 + 1) \times 2 = 142$.)

(b) If the 1^{st} term is 1406, then clearly this is a Dlin sequence that includes 1406.
If the 2^{nd} term in a Dlin sequence is 1406, then the 1^{st} term in the sequence is $\frac{1406}{2} - 1 = 703 - 1 = 702$.
If the 3^{rd} term in a Dlin sequence is 1406, then the 2^{nd} term is 702 (as calculated in the line above) and the 1^{st} term in the sequence is $\frac{702}{2} - 1 = 351 - 1 = 350$.
If the 4^{th} term in a Dlin sequence is 1406, then the 3^{rd} term is 702, the 2^{nd} term is 350, and the 1^{st} term in the sequence is $\frac{350}{2} - 1 = 175 - 1 = 174$.

At this point, we see that 174, 350, 702, and 1406 are possible 1^{st} terms which give a Dlin sequence that includes 1406.
We may continue this process of working backward (dividing by 2 and subtracting 1) to determine all possible 1^{st} terms which give a Dlin sequence that includes 1406.

$$1406 \to 702 \to 350 \to 174 \to \frac{174}{2} - 1 = 86 \to \frac{86}{2} - 1 = 42 \to \frac{42}{2} - 1 = 20 \to \frac{20}{2} - 1 = 9$$

Attempting to continue the process beyond 9 gives $\frac{9}{2} - 1 = \frac{7}{2}$ which is not possible since the 1^{st} term in a Dlin sequence must be a positive integer (and so all terms are positive integers).
Thus, the possible 1^{st} terms which give a Dlin sequence that includes 1406 are 9, 20, 42, 86, 174, 350, 702, and 1406.

(c) Each of the integers from 10 to 19 inclusive is a possible first term, and so we must determine the ones digit of each term which follows each of these ten possible first terms.
If the 1^{st} term is 10, then the 2^{nd} term $(10 + 1) \times 2 = 22$ has ones digit 2, and the 3^{rd} term $(22 + 1) \times 2 = 46$ has ones digit 6.
If the 1^{st} term is 11, then the ones digit of the 2^{nd} term $(11+1) \times 2 = 24$ is 4, and the 3^{rd} term $(24 + 1) \times 2 = 50$ has ones digit 0.
Given each of the possible first terms, we list the ones digits of the 2^{nd} and 3^{rd} terms in the table below.

1^{st} term	10	11	12	13	14	15	16	17	18	19
Units digit of the 2^{nd} term	2	4	6	8	0	2	4	6	8	0
Units digit of the 3^{rd} term	6	0	4	8	2	6	0	4	8	2

From the table above, we see that the only ones digit which repeats itself is 8.
Thus, if the 1^{st} term in the sequence is 18 (has ones digit 8), then the 2^{nd} and 3^{rd} terms in the sequence have ones digit 8 and so all terms will have the same ones digit, 8.
Similarly, if the 1^{st} term in the sequence is 13 (has ones digit 3), then the 2^{nd} and 3^{rd} terms in the sequence have ones digit 8.
It then follows that all further terms after the first will have ones digit 8.
The 1^{st} terms (from 10 to 19 inclusive) which produce a Dlin sequence in which all terms after the 1^{st} term have the same ones digit are 13 and 18.

(d) If the 1^{st} term in a Dlin sequence is x, then the 2^{nd} term is $(x + 1) \times 2 = 2x + 2$, and the 3^{rd} term is $(2x + 2 + 1) \times 2 = (2x + 3) \times 2 = 4x + 6$.
For example, if $x = 1$ (note that this is the smallest possible 1^{st} term of a Dlin sequence), then the 3^{rd} term is $4 \times 1 + 6 = 10$, and if $x = 2$, the 3^{rd} term is $4 \times 2 + 6 = 14$.
What is the largest possible value of x (the 1^{st} term of the sequence) which makes $4x + 6$ (the 3^{rd} term of the sequence) less than or equal to 2020?
Setting $4x + 6$ equal to 2020 and solving, we get $4x = 2014$ and so $x = 503.5$.
Since the 1^{st} term of the sequence must be a positive integer, the 3^{rd} term cannot be 2020.
Similarly, solving $4x + 6 = 2019$, we get that x is not an integer and so the 3^{rd} term of a Dlin sequence cannot equal 2019.
When $4x + 6 = 2018$, we get $4x = 2012$ and so $x = 503$.
Thus, if a Dlin sequence has 1^{st} term equal to 503, then the 3^{rd} term of the sequence is a positive integer between 1 and 2020, namely 2018.
Further, 503 is the largest possible 1^{st} term for which the 3^{rd} term has this property.
Each 1^{st} term x will give a different 3^{rd} term, $4x + 6$.

Thus, to count the number of positive integers between 1 and 2020, inclusive, that can be the 3^{rd} term in a Dlin sequence, we may count the number of 1^{st} terms which give a 3^{rd} term having this property.
The smallest possible 1^{st} term is 1 (giving a 3^{rd} term of 10) and the largest possible 1^{st} term is 503 (which gives a 3^{rd} term of 2018).
Further, every value of x between 1 and 503 gives a different 3^{rd} term between 10 and 2018.
Thus, there are 503 positive integers between 1 and 2020, inclusive, that can be the 3^{rd} term in a Dlin sequence.

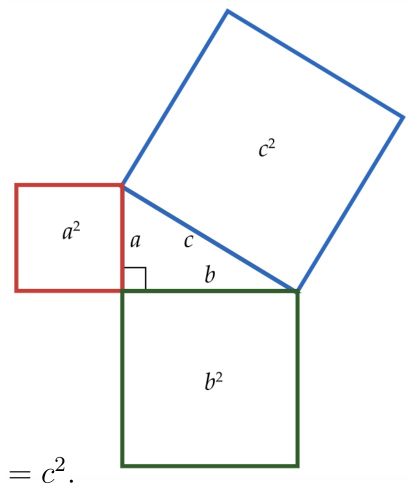
CEMC at Home

Grade 9/10 - Tuesday, June 9, 2020
Pythagorean Triples

The Pythagorean Theorem: In a right-angled triangle, where c represents the length of the hypotenuse and a and b represent the lengths of the two legs (the two shorter sides), the following equation is true:

$$a^2 + b^2 = c^2$$

Note: It is also true that any triangle with side lengths a, b, and c that satisfy the equation $a^2 + b^2 = c^2$ must be a right-angled triangle.



A *Pythagorean triple* is a triple of integers (a, b, c) that satisfies $a^2 + b^2 = c^2$.

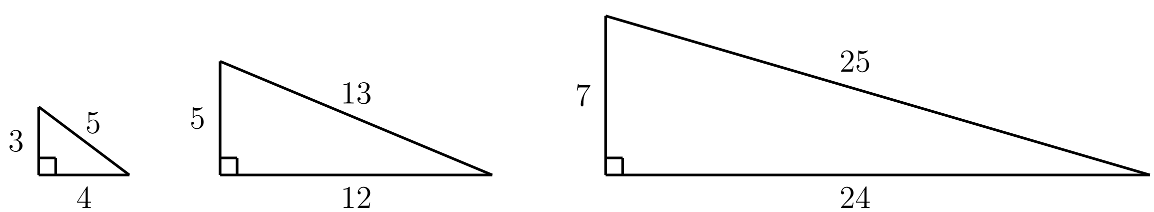
If a triangle is formed with integer side lengths a, b, and c, then this triangle is a right-angled triangle exactly when (a, b, c) is a Pythagorean triple.

Problem 1: The triple $(3, 4, 5)$ is a Pythagorean triple. If $a = 3$, $b = 4$, and $c = 5$, then

$$a^2 + b^2 = 3^2 + 4^2 = 9 + 16 = 25 \quad \text{and} \quad c^2 = 5^2 = 25$$

and therefore, $a^2 + b^2 = c^2$.

Show that $(5, 12, 13)$ and $(7, 24, 25)$ are also Pythagorean triples.



Is every positive integer a part of at least one Pythagorean triple? It turns out that the answer to this question is no. However, every positive integer that is *at least 3* is a part of a Pythagorean triple. Let's explore this idea.

Problem 2: Consider the Pythagorean triples from Problem 1: $(3, 4, 5)$, $(5, 12, 13)$, and $(7, 24, 25)$. Notice that the integers in the leftmost coordinates of the triples are the odd integers 3, 5, and 7. Do you notice a pattern in the other two integers in each triple? The remaining two integers in each triple are consecutive integers: 4 and 5, 12 and 13, 24 and 25. Let's explore this pattern.

(a) Build a Pythagorean triple that includes the odd integer 9 by following these steps:

(i) Determine n such that $9^2 = 2n + 1$. (*Answer:* $n = \frac{81-1}{2} = 40$.)

(ii) Verify that $(n + 1)^2 - n^2 = 9^2$. (*Answer:* $41^2 - 40^2 = 1681 - 1600 = 81 = 9^2$.)

(iii) Write down a Pythagorean triple for which the smallest integer is 9. (*Answer:* Since $41^2 - 40^2 = 9^2$, we have $9^2 + 40^2 = 41^2$ and so $(9, 40, 41)$ is a Pythagorean triple.)

(b) Build a Pythagorean triple that includes the odd integer 11 by following these steps:

(i) Determine n such that $11^2 = 2n + 1$.

(ii) Verify that $(n + 1)^2 - n^2 = 11^2$.

(iii) Write down a Pythagorean triple for which the smallest integer is 11.

(c) Use the ideas from (a) and (b) to build Pythagorean triples that include the next four odd integers: 13, 15, 17, 19.

Can you see how to do this for any odd integer? We explore this in the challenge problem.

Problem 3:

(a) Consider the Pythagorean triple $(3, 4, 5)$. Show that if you multiply each integer in the triple by 2, then you obtain another Pythagorean triple.

(b) Use the idea from (a) to build another Pythagorean triple that includes the odd integer 9.

(c) Show that for every positive integer n, the triple $(3n, 4n, 5n)$ is a Pythagorean triple.

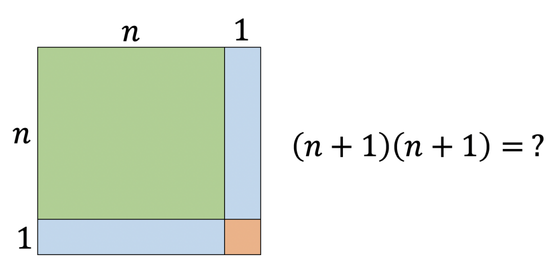
It is also true that $(5n, 12n, 13n)$ *and* $(7n, 24n, 25n)$ *are Pythagorean triples.*

(d) Use the ideas from Problem 2 and Problem 3 to show that every integer from 4 to 20 is part of at least one Pythagorean triple.

Challenge Problem: Think about how you might use some of these ideas to show that every integer that is at least 3 is part of a Pythagorean triple. One possible approach is outlined below, but there are others:

(a) Odd numbers:

(i) Show that for every positive integer n, we have $(n + 1)^2 - n^2 = 2n + 1$.



(ii) Use the identity from part (i) to explain why every odd integer that is at least 3 is part of a Pythagorean triple.

(b) Even numbers:

(i) Show that for every positive integer n, we have $(n + 2)^2 - n^2 = 4n + 4$.

(ii) Use the identity from part (i) to explain why every even integer that is at least 4 is part of a Pythagorean triple.

More Info:

Check out the CEMC at Home webpage on Tuesday, June 16 for a solution to Pythagorean Triples.

CEMC at Home

Grade 9/10 - Tuesday, June 9, 2020
Pythagorean Triples - Solution

Problem 1: The triple $(3, 4, 5)$ is a Pythagorean triple. Show that $(5, 12, 13)$ and $(7, 24, 25)$ are also Pythagorean triples.

Solution:

If $a = 5$, $b = 12$, and $c = 13$, then $a^2 + b^2 = 5^2 + 12^2 = 25 + 144 = 169$ and $c^2 = 13^2 = 169$. Therefore, $a^2 + b^2 = c^2$.

If $a = 7$, $b = 24$, and $c = 25$, then $a^2 + b^2 = 7^2 + 24^2 = 49 + 576 = 625$ and $c^2 = 25^2 = 625$. Therefore, $a^2 + b^2 = c^2$.

Problem 2:

(a) Build a Pythagorean triple that includes the odd integer 9 by following these steps:

  (i) Determine n such that $9^2 = 2n + 1$. (*Answer:* $n = \frac{81-1}{2} = 40$.)

  (ii) Verify that $(n + 1)^2 - n^2 = 9^2$. (*Answer:* $41^2 - 40^2 = 1681 - 1600 = 81 = 9^2$.)

  (iii) Write down a Pythagorean triple for which the smallest integer is 9. (*Answer:* Since $41^2 - 40^2 = 9^2$, we have $9^2 + 40^2 = 41^2$ and so $(9, 40, 41)$ is a Pythagorean triple.)

(b) Build a Pythagorean triple that includes the odd integer 11 by following these steps:

  (i) Determine n such that $11^2 = 2n + 1$.

  (ii) Verify that $(n + 1)^2 - n^2 = 11^2$.

  (iii) Write down a Pythagorean triple for which the smallest integer is 11.

(c) Use the ideas from (a) and (b) to build Pythagorean triples that include the next four odd integers: 13, 15, 17, 19.

Solution:

(b) (i) We have $11^2 = 121 = 2n + 1$ exactly when $n = \frac{121-1}{2} = 60$.

  (ii) When $n = 60$ we have $(n + 1)^2 - n^2 = 61^2 - 60^2 = 3721 - 3600 = 121 = 11^2$.

  (iii) Since $61^2 - 60^2 = 11^2$, we have $11^2 + 60^2 = 61^2$ and so $(11, 60, 61)$ is a Pythagorean triple involving the integer 11.

(c) For the integer 13: We have $13^2 = 169 = 2n + 1$ exactly when $n = \frac{169-1}{2} = 84$. We can verify that $85^2 - 84^2 = 13^2$ which means $13^2 + 84^2 = 85^2$ and so $(13, 84, 85)$ is a Pythagorean triple involving the integer 13.

Using this same method, we can also obtain the following Pythagorean triples:

$$(15, 112, 113), (17, 144, 145), (19, 180, 181)$$

Problem 3:

(a) Consider the Pythagorean triple $(3, 4, 5)$. Show that if you multiply each integer in the triple by 2, then you obtain another Pythagorean triple.

Solution:

If we multiply each integer in the triple $(3, 4, 5)$ by 2, then we obtain the triple $(6, 8, 10)$. We can check that this triple is a Pythagorean triple as follows: $6^2 + 8^2 = 36 + 64 = 100 = 10^2$.

(b) Use the idea from (a) to build another Pythagorean triple that includes the odd integer 9.

Solution:

If we multiply each integer in the triple $(3, 4, 5)$ by 3, then we obtain the triple $(9, 12, 15)$. We can check that this triple is a Pythagorean triple as follows: $9^2 + 12^2 = 81 + 144 = 225 = 15^2$.

Note that we have now found two different Pythagorean triples that involve the odd number 9: $(9, 40, 41)$ *and* $(9, 12, 15)$.

(c) Show that for every positive integer n, the triple $(3n, 4n, 5n)$ is a Pythagorean triple.

It is also true that $(5n, 12n, 13n)$ *and* $(7n, 24n, 25n)$ *are Pythagorean triples.*

Solution:

First we note that for every positive n, the numbers $3n$, $4n$, and $5n$ are positive integers. Also, we have $(3n)^2 + (4n)^2 = 9n^2 + 16n^2 = 25n^2 = (5n)^2$. This means that the triple $(3n, 4n, 5n)$ is a Pythagorean triple.

Note: In a similar way, we can show that $(5n)^2 + (12n)^2 = 25n^2 + 144n^2 = 169n^2 = (13n)^2$ *and* $(7n)^2 + (24n)^2 = 49n^2 + 576n^2 = 625n^2 = (25n)^2$.

(d) Use the ideas from Problem 2 and Problem 3 to show that every integer from 4 to 20 is part of at least one Pythagorean triple.

Solution:

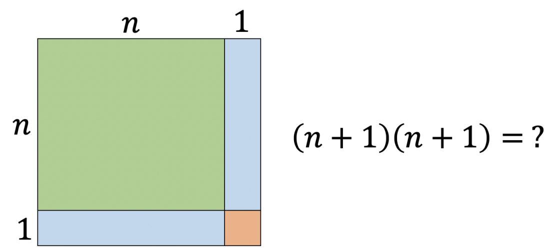
We provide at least one triple for each integer. Some of the triples given below have already been justified earlier. See if you can determine how the other triples were built using the ideas from Problem 2 and Problem 3. For example, the second triple given for 10 was obtained by multiplying each integer in the Pythagorean triple $(5, 12, 13)$ by 2 and using the idea from Problem 3(c).

4: $(3, 4, 5)$
5: $(3, 4, 5)$, $(5, 12, 13)$
6: $(6, 8, 10)$
7: $(7, 24, 25)$
8: $(6, 8, 10)$
9: $(9, 40, 41)$, $(9, 12, 15)$
10: $(6, 8, 10)$, $(10, 24, 26)$
11: $(11, 60, 61)$
12: $(5, 12, 13)$, $(9, 12, 15)$, $(12, 16, 20)$
13: $(13, 84, 85)$
14: $(14, 48, 50)$
15: $(9, 12, 15)$, $(15, 112, 113)$
16: $(12, 16, 20)$
17: $(17, 144, 145)$
18: $(18, 24, 30)$, $(18, 80, 82)$
19: $(19, 180, 181)$
20: $(12, 16, 20)$, $(20, 48, 52)$

Challenge Problem: Think about how you might use some of these ideas to show that every integer that is at least 3 is part of a Pythagorean triple. One possible approach is outlined below, but there are others:

(a) Odd numbers:

(i) Show that for every positive integer n, we have $(n+1)^2 - n^2 = 2n + 1$.



(ii) Use the identity from part (i) to explain why every odd integer that is at least 3 is part of a Pythagorean triple.

Solution:

(i) *Method 1:* Using the image provided, we see that the area of the largest square is represented by the quantity $(n+1)(n+1)$, the area of the medium square is represented by the quantity $(n)(n)$, the area of each of the two rectangles is represented by the quantity $(1)(n)$, and the area of the smallest square is represented by the quantity $(1)(1)$. Since the medium square, small square and two rectangular regions are used to form the larger square we must have

$$(n+1)(n+1) = (n)(n) + (1)(n) + (1)(n) + (1)(1)$$

This simplifies to $(n+1)^2 = n^2+2n+1$ which can be rearranged to give $(n+1)^2-n^2 = 2n+1$.
Method 2: Using the distributive property, we have $(n+1)(n+1) = (n+1)(n)+(n+1)(1)$ which means

$$(n+1)^2 = (n+1)(n+1) = (n+1)(n) + (n+1)(1) = n^2 + n + n + 1 = n^2 + 2n + 1$$

It follows that $(n+1)^2 - n^2 = (n^2 + 2n + 1) - n^2 = 2n + 1$.
Method 3: If you have seen the formula for the difference of squares before, then you may see that

$$(n+1)^2 - n^2 = ((n+1) + n)((n+1) - n) = (2n+1)(1) = 2n + 1$$

(ii) We follow the method from Problem 2 for a general odd integer k that is *greater than* 1: Let $n = \frac{k^2-1}{2}$. Since k^2 must also be an odd integer that is greater than 1, $k^2 - 1$ must be an even integer that is greater than 0. This means n is a positive integer. Rearranging the equation gives $k^2 = 2n + 1$. Using the formula from (i), we get that

$$(n+1)^2 - n^2 = 2n + 1 = k^2$$

for this value of n. Rearranging the equation above gives

$$k^2 + n^2 = (n+1)^2$$

which shows that $(k, n, n+1)$ is a Pythagorean triple that includes the given odd integer k.

(b) Even numbers:

(i) Show that for every positive integer n, we have $(n+2)^2 - n^2 = 4n + 4$.

(ii) Use the identity from part (i) to explain why every even integer that is at least 4 is part of a Pythagorean triple.

Solution:

(i) Since $(n+2)^2 = (n+2)(n+2) = (n+2)(n) + (n+2)(2) = n^2 + 2n + 2n + 4 = n^2 + 4n + 4$ we have $(n+2)^2 - n^2 = (n^2 + 4n + 4) - n^2 = 4n + 4$.

(ii) To show this you can follow the method from part (a) of the challenge problem. We do not give a full solution here, but instead outline the steps using an example.

We can build a Pythagorean triple that includes the even integer 6 by following these steps:

- Determine n such that $6^2 = 4n + 4$:
  Solving we get $n = \frac{36-4}{4} = 8$.
- From part (i) above, we know that for this n we will have $(n+2)^2 - n^2 = 6^2$.
  We can verify this directly: $(8+2)^2 - 8^2 = 10^2 - 8^2 = 100 - 64 = 36 = 6^2$.
- This works shows that $(6, 8, 10)$ is a Pythagorean triple.

Suppose that k is an even integer that is *greater than* 2. If you can find find a positive integer n such that $k^2 = 4n + 4$, then the steps above show that $(k, n, n+2)$ is a Pythagorean triple. Can you see why there will always be such a value of n?

If $k > 2$ and is even then $k^2 > 4$ and is a multiple of 4, and so $k^2 - 4 > 0$ and is a multiple of 4. It follows that $n = \frac{k^2-4}{4}$ is a positive integer and satisfies $k^2 = 4k + 4$ as needed!

Parts (a) and (b) of the challenge problem show that every integer that is at least 3 is part of a Pythagorean triple. Can you explain why the integers 1 and 2 cannot be part of Pythagorean triples?

CEMC at Home

Grade 9/10 - Wednesday, June 10, 2020
Interplanetary Bases

You are part of an interplanetary mission to catalogue the number of elements present on two of Jupiter's moons. Six different astronauts have reported the number of elements they discovered, but to play a prank on you, they reported their findings using different *number systems* than you are used to. Making matters worse, they refuse to tell you exactly what number systems they have used.

Astronaut	**Total Number of Elements Discovered**	**Base used**	**Number of Elements Discovered on Moon 1**	**Number of Elements Discovered on Moon 2**
Afon	131			
Breanna	105			
Cheng	221			
Denisa	56			
Eka	105			
Fergus	221			

Each of the astronauts reported their numbers using a number system with some *base* b.

You normally use a *base 10* number system. In the base 10 system, you can use any of the digits $0, 1, 2, 3, 4, 5, 6, 7, 8$, or 9 to form the digits in your numbers. In this base, the numeral 763 represents the integer with value

$$7 \times 100 + 6 \times 10 + 3 \times 1 = 7 \times 10^2 + 6 \times 10^1 + 3 \times 10^0$$

Each astronaut used a number system with a positive integer b as the base, with $4 \leq b \leq 7$. In the base b system, they can use any of the digits $0, 1, \ldots, (b-1)$ to form the digits in their numbers. In this base, the numeral ABC (where A, B, and C are digits) represents the integer with value

$$A \times b^2 + B \times b + C \times 1 = A \times b^2 + B \times b^1 + C \times b^0$$

For example, in the system with base $b = 4$, the numeral 203 represents the integer with value

$$2 \times 4^2 + 0 \times 4 + 3 \times 1$$

but the numerals 124 and 552 have no meaning in this system since they contain digits larger than 3.

Problem 1: The astronauts gave you two clues about the overall report:

- Exactly two of the astronauts reported their numbers in base $b = 4$.
- All six astronauts discovered (and reported discovering) the same number of elements in total.

Determine how many elements were discovered on the two moons, in total (giving your answer in the usual base 10 system), and determine which base each of the astronauts was using to report their findings.

Problem 2: You are now told that each astronaut discovered 16 distinct elements on Moon 1 and discovered 3 elements common to both moons, with these numbers given in base 10. Using this information, fill in the blank cells in the table above. *Make sure to write the numbers in the right base!*

More Info:

Check out the CEMC at Home webpage on Wednesday, June 17 for a solution to Interplanetary Bases.

CEMC at Home

Grade 9/10 - Wednesday, June 10, 2020
Interplanetary Bases - Solution

Set-up: Six different astronauts have reported the number of elements they discovered on two moons, but reported their findings using different *number systems* than you are used to.

Astronaut	Total Number of Elements Discovered	Base used	Number of Elements Discovered on Moon 1	Number of Elements Discovered on Moon 2
Afon	131			
Breanna	105			
Cheng	221			
Denisa	56			
Eka	105			
Fergus	221			

Each astronaut used a number system with a positive integer b as the base, with $4 \leq b \leq 7$. In the base b system, they can use any of the digits $0, 1, \ldots, (b-1)$ to form the digits in their numbers. In this base, the numeral ABC (where A, B, and C are digits) represents the integer with value

$$A \times b^2 + B \times b + C \times 1 = A \times b^2 + B \times b^1 + C \times b^0$$

Problem 1: The astronauts gave you two clues about the overall report:

- Exactly two of the astronauts reported their numbers in base $b = 4$.
- All six astronauts discovered (and reported discovering) the same number of elements in total.

Determine how many elements were discovered on the two moons, in total, and determine which base each of the astronauts was using to report their findings.

Solution:

If two astronauts reported their findings using the same number system, then they will have reported identical numerals. The only numerals that appear more than once in the table above are 105 and 221. This means one of these two numerals must represent the total number of elements discovered written in base $b = 4$. Since a base 4 representation of a number can only use the digits 0, 1, 2, and 3, the numeral 105 cannot be a base 4 representation of any number. This means the total number of elements discovered is represented in base 4 as 221. In base 4, the numeral 221 represents the integer $2 \times 4^2 + 2 \times 4 + 1 = 32 + 8 + 1 = 41$.

We now know that Cheng and Fergus reported their findings in base 4, and that there were 41 elements discovered in total. There are several ways to approach the problem from here.

Consider the numeral 131. We know this is not in base 4 because the total in base 4 is represented by 221. The largest digit in 131 is 3, so this numeral makes sense in all three other bases: 5, 6, and 7. In base 5, it represents the integer $1 \times 5^2 + 3 \times 5^1 + 1 = 25 + 15 + 1 = 41$. Notice that in base 6, the numeral 131 would represent the integer $1 \times 6^2 + 3 \times 6 + 1 = 36 + 18 + 1 = 55$ and in base 7 it would represent $7^2 + 3 \times 7 + 1 = 71$. We conclude that the numeral 131 is in base 5 and that Afon used base 5.

This means the numeral 105 represents the total in either base 6 or base 7. In base 6, it represents the integer $1 \times 6^2 + 0 \times 6 + 5 = 36 + 5 = 41$. In base 7, it represents the integer $1 \times 7^2 + 0 \times 7 + 5 = 54$. We conclude that the numeral 105 is in base 6 and so Breanna and Eka used base 6.

Finally, we suspect that 56 is in base 7, and indeed, $5 \times 7 + 6 = 41$, so we conclude that Denisa reported in base 7.

We can also solve this problem using algebra once we have determined that the total number of elements is 41. For example, we know that the numeral 56 is the base b representation of 41 for some b between 4 and 7 inclusive. This means $41 = 5b + 6$, or $35 = 5b$, which can be solved for b to get $b = 7$. Similarly, to determine in which base the numeral 131 represents the number 41, we can solve the equation $41 = 1 \times b^2 + 3 \times b + 1$ for b. Rearranging, we get $40 = b^2 + 3b$. If you have experience factoring quadratics, you can solve this equation for b. If not, since you only have four possibilities for b, you can check $b = 4, 5, 6, 7$ and find that $b = 5$ is the only solution among these four choices. Finally, the numeral reported by Breanna leads to the equation $41 = b^2 + 5$ or $36 = b^2$. Since b is positive, this means $b = 6$.

Problem 2: You are now told that each astronaut discovered 16 distinct elements on Moon 1 and discovered 3 elements common to both moons, with these numbers given in base 10. Using this information, fill in the blank cells in the table above.

Solution:

The number of elements that were discovered on only Moon 1 is $16 - 3 = 13$, which means there must have been 28 elements discovered on Moon 2. Filling in the table means finding the base 4, 5, 6, and 7 representations of 16 and 28. These representations are given in the table below. To help explain how these numerals were obtained, we show the work for the first row below the table.

Astronaut	Total Number of Elements Discovered	Base used	Number of Elements Discovered on Moon 1	Number of Elements Discovered on Moon 2
Afon	131	5	31	103
Breanna	105	6	24	44
Cheng	221	4	34	130
Denisa	56	7	22	40
Eka	105	6	24	44
Fergus	221	4	34	130

To represent the integer 16 in base 5, we first note that its base 5 representation must have at most two digits. This is because the numeral ABC represents the integer with value $A \times 5^2 + B \times 5 + C$ which is at least 25 (assuming A is a positive digit). Thus, we seek digits A and B between 0 and 4 inclusive so that $A \times 5 + B = 16$. It is easy to check that digits $A = 3$ and $B = 1$ satisfy this equation, and in fact, it is true that no other pair of integers between 0 and 4 inclusive satisfies the equation. Therefore, the numeral 31 is the base 5 representation of the integer 16.

To represent the integer 28 in base 5, notice that 5^3 and hence all larger powers of 5 are greater than 28, so the base 5 representation of 28 has at most three digits. As well, the two-digit numeral with the largest value in base 5 is 44 which represents the integer $4 \times 5 + 4 = 24$, so this means the base 5 representation of 28 has exactly three digits. Therefore, we seek integers A, B, and C all between 0 and 4 inclusive satisfying $A \times 5^2 + B \times 5^1 + C \times 5^0 = 28$ or $25A + 5B + C = 28$. We know that $A \geq 1$ since the representation has three digits, but if $A \geq 2$, then $25A \geq 50 > 28$, so this means we must have $A = 1$. The equation then simplifies to $5B + C = 3$, and the only solution to this equation where B and C are integers between 0 and 4 inclusive is $B = 0$ and $C = 3$. Therefore, the base 5 representation of the integer 28 is 103.

There are systematic ways of expressing numbers in various bases. You may wish to do an internet search to learn more about this.

CEMC at Home

Grade 9/10 - Thursday, June 11, 2020
PRODUCTivity

The *"digit product"* of a positive integer is the product of the individual digits of the integer.

For example, the digit product of 234 is $2 \times 3 \times 4 = 24$. Other numbers also have a digit product of 24. For example, 2223, 113 181 and 38 each have a digit product of 24. The number 38 is the smallest positive integer with a digit product of 24.

There are many positive integers whose digit product is 2000.

Determine the smallest positive integer whose digit product is 2000.

More Info:

Check out the CEMC at Home webpage on Friday, June 12 for a solution to PRODUCTivity.

This CEMC at Home resource is a past problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students during the school year. POTW is wrapped up for the current school year and will resume on September 17, 2020. To subscribe to POTW and to find more past problems and their solutions visit:
https://www.cemc.uwaterloo.ca/resources/potw.php

CEMC at Home

Grade 9/10 - Thursday, June 11, 2020
PRODUCTivity - Solution

Problem:

The *"digit product"* of a positive integer is the product of the individual digits of the integer.

For example, the digit product of 234 is $2 \times 3 \times 4 = 24$. Other numbers also have a digit product of 24. For example, 2223, 113 181 and 38 each have a digit product of 24. The number 38 is the smallest positive integer with a digit product of 24.

There are many positive integers whose digit product is 2000.

Determine the smallest positive integer whose digit product is 2000.

Solution:

Let N be the smallest positive integer whose digit product is 2000.

In order to find N, we must find the minimum possible number of digits whose product is 2000. This is because if the integer a has more digits than the integer b, then $a > b$.
Once we have determined the digits that form N, then the integer N is formed by writing those digits in increasing order.

Note that the digits of N cannot include 0, or else the digit product of N would be 0.
Also, the digits of N cannot include 1, otherwise we could remove the 1 and obtain an integer with fewer digits (and thus, a smaller integer) with the same digit product. Therefore, the digits of N will be between 2 and 9, inclusive.

Since the digit product of N is 2000, we will use the prime factorization of 2000 to help determine the digits of N:

$$2000 = 2^4 \times 5^3$$

In order for a digit to have a factor of 5, the digit must equal 5. Therefore, three of the digits of N are 5.

The remaining digits of N must have a product of $2^4 = 16$. We need to find a combination of the smallest number of digits whose product is 16. We cannot have one digit whose product is 16, but we can have two digits whose product is 16. In particular, $16 = 2 \times 8$ and $16 = 4 \times 4$.

Therefore, N has 5 digits. They are 5, 5, 5, 2, 8 or 5, 5, 5, 4, 4. In order for N to be as small as possible, its digits must be in increasing order. The smallest positive integer formed by the digits 5, 5, 5, 2, 8 is 25 558. The smallest positive integer formed by the digits 5, 5, 5, 4, 4 is 44 555.

Since $25\ 558 < 44\ 555$, the smallest N is 25 558. That is, the smallest positive integer with a digit product of 2000 is 25 558.

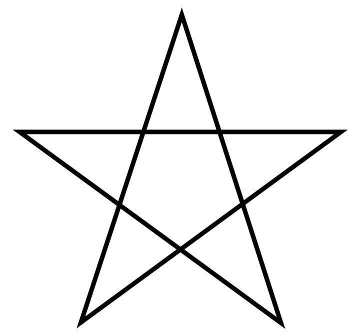
CEMC at Home

Grade 9/10 - Friday, June 12, 2020
Pieces of a Star

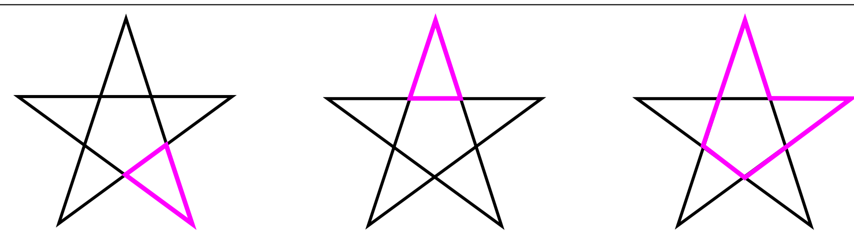
A *polygon* is a two-dimensional closed figure formed by line segments. A *simple polygon* is a polygon that does not "intersect itself". We can think about tracing out the boundary of a simple polygon: we only travel in straight lines and the only time we return to a point for a second time is when we have finished tracing out the entire boundary.

Problem:

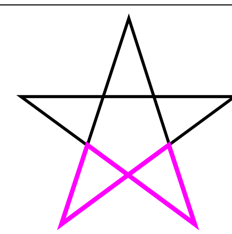
There are 37 simple polygons to be found in the 5-pointed star shown on the right. Describe the 37 simple polygons and explain why there are no more.



The sides of your polygons should all be line segments appearing in the star. To get you started, below are three examples of simple polygons that can be found in the star, along with a polygon that is not simple.



Three simple polygons
(allowed)



Non-simple polygon
(not allowed)

Hints for counting polygons:

1. What simple polygons can you find that are not congruent to any of the examples shown above?
2. How many simple polygons do not include the interior of the pentagon?
3. The number of simple polygons that include the interior of the pentagon is a power of 2.

Follow-up Discussion:

Were you able to find a systematic way of counting the simple polygons in the 5-pointed star? Could this same strategy be used to count the simple polygons in a 6-pointed star or a 7-pointed star?

In general, an n-pointed star, for an integer $n \geq 5$, consists of an n-gon in the centre, with n triangles attached to its sides pointing outwards. Think about the following:

Follow-up Question: How many simple polygons are there in an n-pointed star?

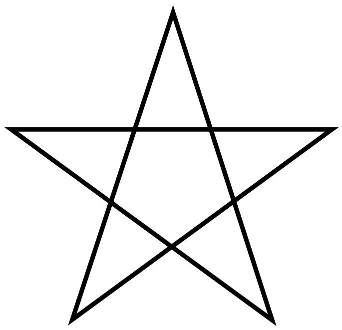
More Info:

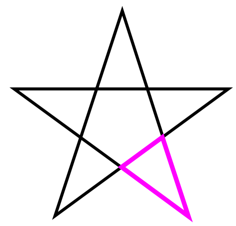
Check out the CEMC at Home webpage on Friday, June 19 for a solution to Pieces of a Star.

CEMC at Home

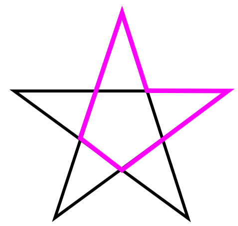
Grade 9/10 - Friday, June 12, 2020
Pieces of a Star - Solution

Problem: There are 37 simple polygons to be found in the 5-pointed star shown. Describe the 37 simple polygons and explain why there are no more.





Example 1



Example 2

Solution: First, we note that there are 5 simple polygons in the star that do not include the pentagon in the centre: the 5 triangles that form the 5 points of the star. One such triangle is shown in Example 1 above.

Every other simple polygon in the star will consist of the pentagon in the centre plus some of the 5 triangles which form the points of the star. One of these polygons is formed by choosing *none* of the triangles. In this case we get the pentagon in the centre. Another of these polygons is formed by choosing *all* of the triangles. In this case we get the entire 5-pointed star (or its boundary). Another of these polygons is formed by choosing the two triangles at the top right. In this case we get the pentagon shown in Example 2.

We could try to systematically draw all of these polygons, but we can count them without doing so. Each such polygon either includes a particular triangle or it does not. This means when drawing one of these polygons, we have two choices for each triangle: in or out. This leads to $2^5 = 32$ different possibilities for which triangles we include, and hence there must be 32 simple polygons that include the pentagon in the centre.

Putting this all together, there are 5 simple polygons that do not include the pentagon (the 5 outer triangles) and 32 simple polygons that include the pentagon. Therefore, there are $5 + 32 = 37$ simple polygons in the star, in total.

Follow-up Question: How many simple polygons are there in an n-pointed star?

Solution: We can think of an n-pointed star as an n-gon with n triangles around it. The n triangles form the n points of the star.

Some of the simple polygons in the star are the n triangles that form the n points of the star. Every other simple polygon will consist of the n-gon in the centre plus some of the n triangles that form the points of the star. Since there are n triangles, and for each triangle we have 2 choices (include or not include), there are 2^n such polygons.

Therefore, in total, we have $n + 2^n$ simple polygons in the n-pointed star.

In particular, when $n = 6$ we have $6 + 2^6 = 70$ and so there are 70 simple polygons in the 6-pointed star, and when $n = 7$ we have $7 + 2^7 = 135$ and so there are 135 simple polygons in the 7-pointed star.