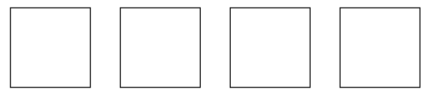
CEMC at Home

Grade 4/5/6 - Monday, April 6, 2020
How Great Is Your Number?

You Will Need:

- At least two players
- A sheet of paper and a pencil for each player
- At least one pair of dice

How To Play:



1. Each player draws four squares on their paper.
2. Players take turns rolling a pair of dice.
3. On your turn, roll the pair of dice and announce the sum of the two numbers rolled.
4. Each player then places the ones digit of this sum in one of the empty squares on their paper.
   For example, if this is the outcome of the roll,

    +  = 12

   then each player places a "2" in one of the empty squares on their paper. If there is more than one empty square remaining, then they have a choice for where to place the "2".
5. Continue taking turns until all squares are filled.
   If there are two players, then each player will roll the dice twice.
6. Each player ends up with a four-digit number. The winner of the game is the player who has formed the greatest four-digit number. (It is possible for the game to end in a tie.)

Play this game a number of times and think about the following questions.

1. What is the least possible ones digit of the sum that you could get on your turn? What is the greatest possible ones digit you could get?
2. If you get a small ones digit, in which square should you place it? Why?
3. If you get a large ones digit, in which square should you place it? Why?
4. Which are the least likely ones digits to occur in this game? Which are the most likely ones digits to occur in this game?

Variations:

A. Try the same game but with three squares, and then with five squares.

B. Try the same game but take the ones digit of the product of the two numbers rolled on the dice, instead of the sum. How does this change the answers to the earlier questions?

More info:

Check out the CEMC at Home webpage on Tuesday, April 14 for answers to these questions.

CEMC at Home

Grade 4/5/6 - Monday, April 6, 2020
How Great Is Your Number? - Solution

Play the game a number of times and think about the following questions.

1. What is the least possible ones digit of the sum that you could get on your turn? What is the greatest possible ones digit you could get?

   Solution: The least possible ones digit is 0. This comes from a sum of 10, which happens if you roll a pair of fives, or a six on one die and a four on the other.

   The greatest possible ones digit is 9. This comes from a sum of 9, which happens if you roll a six on one die and a three on the other, or a five on one die and a four on another.

2. If you get a small ones digit, in which square should you place it? Why?

   Solution: For the four-digit number, the four squares from left to right are the thousands, hundreds, tens, and ones digit, respectively. Since the winner is the player with the greatest four-digit number, small digits should be placed as far as possible to the right. For example, if a digit of 1 comes up, it would make sense to place the digit 1 in an empty square that is farthest to the right.

   How should you you deal with a ones digit of 0 if it comes up? How should you deal with numbers that are smaller than average, but not as small as 1 (for example 3 or 4)?

3. If you get a large ones digit, in which square should you place it? Why?

   Solution: Large digits should be placed as far as possible to the left. For example, if a digit of 9 comes up, it would make sense to place the digit 9 in an empty square that is farthest to the left.

   How should you deal with digits that are larger than average, but not as large as 9? If a ones digit of 6 or 7 comes up first, then where should you place it? You might decide to place the digit 7 in the leftmost square, and then have a ones digit of 9 come up next!

4. Which are the least likely ones digits to occur in this game? Which are the most likely ones digits to occur in this game?

   Solution: Label the two dice "Die #1" and "Die #2". There are 36 possible possible rolls and the chances of getting each of these rolls is the same. Below is a table showing the sum of the two dice in each of the 36 possible rolls.

		Die #1					
		1	2	3	4	5	6
Die #2	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

Notice that there are two rolls which yield a sum with a units digit of 1: a six on Die #1 and a 5 on Die #2, or a five on Die #1 and a 6 on Die #2, giving $6 + 5 = 5 + 6 = 11$. Similarly, a ones digit of 2 can happen in two ways, $1 + 1 = 2$ or $6 + 6 = 12$, and a ones digit of 3 can happen in two ways, $1 + 2 = 3$ or $2 + 1 = 3$. We can see from the table that the ones digits of 1, 2, and 3 are the least likely to occur in the game (with two ways to achieve each digit).

Similar reasoning shows that there are

- three ways to get a ones digit of 4 ($2 + 2 = 4$, $1 + 3 = 4$, $3 + 1 = 4$),
- three ways to get a ones digits of 0 ($5 + 5 = 10$, $6 + 4 = 10$, $4 + 6 = 10$),
- four ways to get a ones digit of 5 ($1 + 4$, $4 + 1$, $2 + 3$, $3 + 2$),
- four ways to get a ones digit of 9 ($3 + 6$, $6 + 3$, $4 + 5$, $5 + 4$),
- five ways to get a ones digit of 6 ($1 + 5$, $5 + 1$, $2 + 4$, $4 + 2$, $3 + 3$),
- five ways to get a ones digit of 8 ($2 + 6$, $6 + 2$, $3 + 5$, $5 + 3$, $4 + 4$), and
- six ways to get a ones digit of 7 ($1 + 6$, $6 + 1$, $2 + 5$, $5 + 2$, $3 + 4$, $4 + 3$).

Thus the most likely ones digit is 7 (with six ways to achieve this digit).

Variations:

A. Try the same game but with three squares, and then with five squares.

These games will be similar, but will end after a different number of rolls.

B. Try the same game but take the ones digit of the product of the two numbers rolled on the dice, instead of the sum. How does this change the answers to the earlier questions?

The chances of getting certain ones digits change in this variation. For example, the most likely ones digits to occur in this version are 0, 2, and 6, and the least likely ones digit to occur is 7, as it cannot occur at all! Otherwise, the basic strategy of the game remains the same.

CEMC at Home

Grade 4/5/6 - Tuesday, April 7, 2020

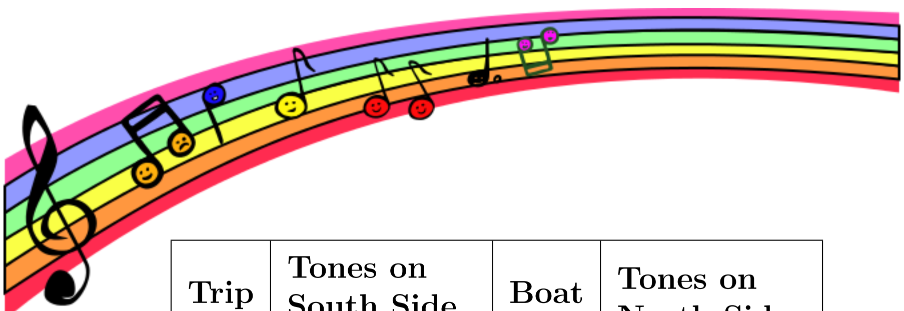
Carrying a Tune

While exploring in the woods, you have found and captured five *Pure Tones*: magical objects that each produce a single, pure musical note. You have put these Tones in glass jars labelled 1, 2, 3, 4, and 5, organized from lowest note to highest note.

In order to take these Tones home, you have to transport them across a river, from the south side to the north side. However, your boat only has storage space for two Tones at a time, plus a seat for you, the driver.

The problem is that these Tones only stay quiet while you are watching them. If they are left alone on one side of the river, they will start making noise. If Tones that are one note apart are left together (like 1 and 2, or 4 and 5), their combined noise will shatter their glass jars, and they will escape.

Design a set of trips back and forth across the river so that you and the five Tones end up on the north side together, without any of them escaping. The table below may help organize your thinking.



Trip	**Tones on South Side**	**Boat**	**Tones on North Side**
	1, 2, 3, 4, 5		None
1		→	
2		←	
3		→	
4		←	
5		→	
6		←	
⋮			

Challenges:

- Can you still solve the puzzle if you can never carry *only one* Tone on your boat?
- This problem is an example of a "river-crossing puzzle". These are puzzles where the goal is to get from one side of a river to another, subject to various constraints. Design your own river-crossing puzzle, solve it, and share it with a friend or family member.

More Info:

Check out the CEMC at Home webpage on Tuesday, April 14 for a solution to Carrying a Tune.

CEMC at Home

Grade 4/5/6 - Tuesday, April 7, 2020
Carrying a Tune - Solution

Problem: While exploring in the woods, you have found and captured five *Pure Tones*: magical objects that each produce a single, pure musical note. You have put these Tones in glass jars labelled 1, 2, 3, 4, and 5, organized from lowest note to highest note.

In order to take these Tones home, you have to transport them across a river, from the south side to the north side. However, your boat only has storage space for two Tones at a time, plus a seat for you, the driver. The problem is that these Tones only stay quiet while you are watching them. If they are left alone on one side of the river, they will start making noise. If Tones that are one note apart are left together (like 1 and 2, or 4 and 5), their combined noise will shatter their glass jars, and they will escape.

Design a set of trips back and forth across the river so that you and the five Tones end up on the north side together, without any of them escaping.

Solution: The first important thing to notice is that you *must* take Tones 2 and 4 across the river on the first trip. If you do not, then the jars on the south side will shatter during your first trip! (Can you see why?) You are safe to leave Tones 2 and 4 on the north side and head back to the south side on your own. Now you have some choice in which Tones you take across the river on your next trip. You can take any two of the three Tones: 1, 3, and 5. (In our example in the table below, we take Tones 1 and 5.) No matter what choice is made, the next important thing to notice is that you *must* take at least one Tone back from the north to the south side of the river once four Tones have made it to the north side. (In our example, we cannot leave tones 1, 2, 4, and 5 together.) You actually must take two Tones back with you. (In our example, we would have to take either 1 or 2 and either 4 or 5 back to the south side on our next trip. We take Tones 2 and 4.)

The table below shows one possible way to get the Tones across successfully in exactly seven trips.

Trip	Tones on South Side	Boat	Tones on North Side
	1, 2, 3, 4, 5		None
1	1, 3, 5	$2, 4 \rightarrow$	2, 4
2	1, 3, 5	$\leftarrow$	2, 4
3	3	$1, 5 \rightarrow$	1, 2, 4, 5
4	2, 3, 4	$\leftarrow 2, 4$	1, 5
5	2, 4	$3 \rightarrow$	1, 3, 5
6	2, 4	$\leftarrow$	1, 3, 5
7	None	$2, 4 \rightarrow$	1, 2, 3, 4, 5

Challenge: Can you still solve the puzzle if you can never carry *only one* Tone on your boat?

Solution: No, it is impossible to get all five Tones across the river under these conditions. If you can only move zero Tones or two Tones across the river on each trip, then the number of Tones on the north side at any given time will always be an even number! Since we need to end up with an odd number of Tones (five) on the north side, we cannot possibly succeed.

CEMC at Home

Grade 4/5/6 - Wednesday, April 8, 2020
A Stitch in Time...

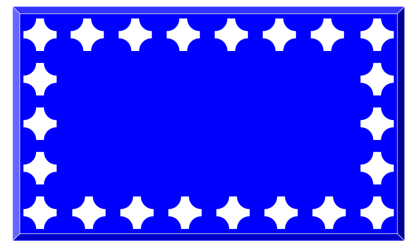
Friends Nathan and Nia have decided to get together to complete a sewing project. They are making fabric placemats with a fancy pattern on the border.

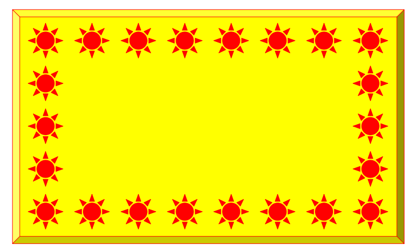
Nathan's pattern is for a blue placemat with a white border; each one takes 7 minutes to complete.

Nia's pattern is a more complicated design for a yellow placemat with a red border; each one takes 9 minutes to complete.

They set up their sewing machines on opposite ends of a table. As they finish each placemat, they put it on top of a pile in the middle of the table.

They both start sewing at the same time. So Nathan puts the first blue placemat at the bottom of the pile after 7 minutes, and Nia puts a yellow placemat on top 2 minutes later, or 9 minutes after they started.





1. Working upwards, complete the table that shows the order of the placemats in the pile after they have been working for 40 minutes. Enter the time at which each placemat is added to the pile as shown.

2. How many minutes after they start sewing will Nathan try to add a blue placemat to the pile at the same time as Nia tries to add a yellow one?

   We are looking for the first time this happens.

3. Suppose instead that a blue placemat takes 6 minutes to make, and a yellow placemat takes 10 minutes. How many minutes after they start sewing will Nathan and Nia try to add their placemats to the pile at the same time?

   We are looking for the first time this happens.

Time	Table Top
14	blue
9	yellow
7	blue

More Info:

Check out the CEMC at Home webpage on Wednesday, April 15 for a solution to A Stitch in Time.

CEMC at Home

Grade 4/5/6 - Wednesday, April 8, 2020
A Stitch in Time - Solution

Problem Summary:

Nathan's pattern is for a blue placemat; each one takes 7 minutes to complete. Nia's pattern is for a yellow placemat; each one takes 9 minutes to complete. When they finish one of their placemats, they put it on top of a single pile in the middle of the table. They both start sewing at the same time.

1. Working upwards, complete the table that shows the order of the placemats in the pile after they have been working for 40 minutes. Enter the time at which each placemat is added to the pile as shown.

2. How many minutes after they start sewing will Nathan try to add a blue placemat to the pile at the same time as Nia tries to add a yellow one?

   We are looking for the first time this happens.

3. Suppose instead that a blue placemat takes 6 minutes to make, and a yellow placemat takes 10 minutes. How many minutes after they start sewing will Nathan and Nia try to add their placemats to the pile at the same time?

   We are looking for the first time this happens.

Solution:

1. See the completed table at the right, showing the times at which each placemat is completed.

2. Note that Nathan's blue placemats are added to the pile at times that are multiples of 7, whereas Nia's yellow placemats are added to the pile at times that are multiples of 9. This means that they will both try to add a placemat at the same time if that time is a multiple of both 7 and 9. The least number that is a multiple of both 7 and 9 is 63. This tells us that the first time Nathan and Nia will try to add placemats to the pile at the same time is 63 minutes after they start.

   Note: 63 is the *least common multiple* of 7 and 9.

3. With this new timing, the first time Nathan and Nia will try to add placemats to the pile at the same time is 30 minutes after they start. This is because 30 is the least number that is a multiple of both 6 and 10.

   Note: 30 is the *least common multiple* of 6 and 10.

Time	Table Top
49	
45	
42	
36	yellow
35	blue
28	blue
27	yellow
21	blue
18	yellow
14	blue
9	yellow
7	blue

Notice that the answer for 2. is equal to the product of the two times, 7 and 9, but this is not the case for the answer in 3. The product of 6 and 10 is 60, and Nathan and Nia will both try and put placemats on the pile after 60 minutes, but this is not the first time this will happen.

Computer Science Connections

The main work of a computer is handled by its CPU (**C**entral **P**rocessing **U**nit). Over time, CPUs have become smaller and more powerful. Most computers today have *multi-core processors*, which means there are multiple cores (processing units) on a single computer chip in the machine. These cores can work in parallel (i.e., at the same time on different tasks) to improve the speed and performance of the computer.

Although it is not too difficult with today's technology to create these multi-core chips, it turns out that is quite difficult to take full advantage of their parallel processing potential. One issue that limits the effectiveness of parallel processing is that it can be difficult for multiple cores to share resources. In this problem, we can think of Nathan and Nia as separate cores and the pile of placemats in the middle of the table as a shared resource. They will have to coordinate things when both want to add a placemat on the pile at the same time.

CEMC at Home features Problem of the Week

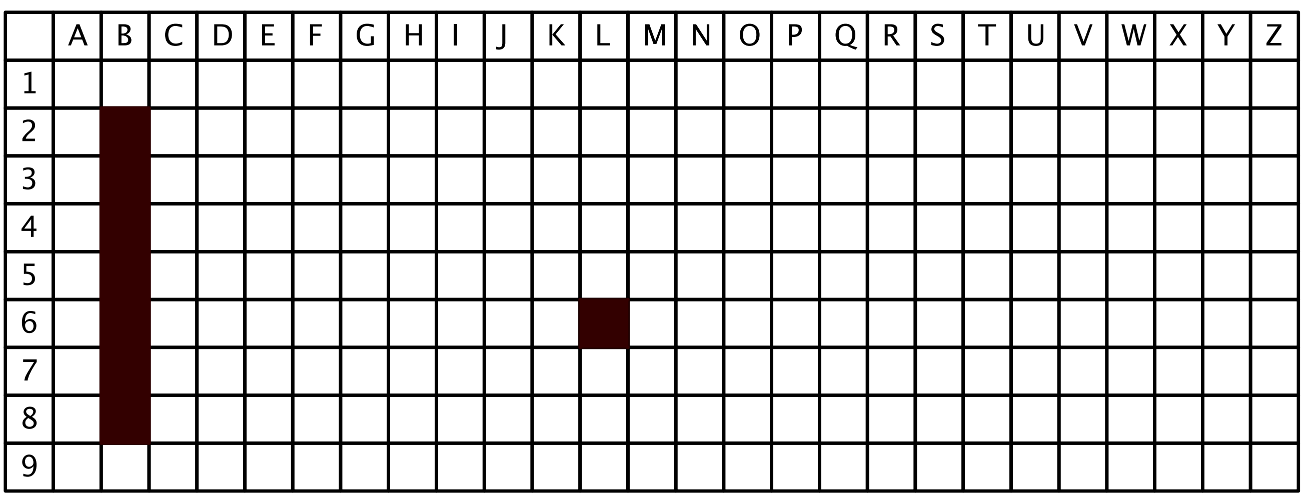
Grade 4/5/6 - Thursday, April 9, 2020

Mystery Code

James wants to send an image to a friend. The image is made up of filled in black squares on a grid. The rows of the grid are labelled with numbers from 1 to 9 and the columns of the grid are labelled from A to Z. James encodes the image by providing a series of short codes. Each short code consists of four parts:

- A letter between A and Z followed by a number between 1 and 9 indicating the column and row of the first square to be filled in on the grid,
- followed by an arrow ($\uparrow, \downarrow, \leftarrow, \rightarrow$) indicating the direction to fill,
- followed by a number, indicating the total number of squares to colour.

For example, the short code $B2 \downarrow 7$ tells you to fill in a square at position $B2$ on the grid, and continue filling in 6 more squares directly below that position. And the short code $L6 \rightarrow 1$ tells you to fill in a square at position $L6$ on the grid and no other squares. This code would have you fill in the grid as shown:



Using the blank grid on the next page, determine the image that James sent using the following codes:

$T8 \uparrow 7$ $\quad D4 \leftarrow 1$ $\quad R2 \leftarrow 5$ $\quad I5 \rightarrow 3$ $\quad F2 \downarrow 7$ $\quad L8 \uparrow 6$ $\quad C3 \rightarrow 1$

$P3 \downarrow 6$ $\quad X2 \downarrow 7$ $\quad E3 \uparrow 1$ $\quad W5 \leftarrow 3$ $\quad H8 \uparrow 6$ $\quad K2 \leftarrow 3$ $\quad B2 \downarrow 7$

Use the first blank grid to create the image that James sent using the following codes:

$T8 \uparrow 7$ $D4 \leftarrow 1$ $R2 \leftarrow 5$ $I5 \rightarrow 3$ $F2 \downarrow 7$ $L8 \uparrow 6$ $C3 \rightarrow 1$

$P3 \downarrow 6$ $X2 \downarrow 7$ $E3 \uparrow 1$ $W5 \leftarrow 3$ $H8 \uparrow 6$ $K2 \leftarrow 3$ $B2 \downarrow 7$

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
1																										
2																										
3																										
4																										
5																										
6																										
7																										
8																										
9																										

An extra blank grid is provided below. Make your own image and provide a classmate with the code that would be used to create your image.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
1																										
2																										
3																										
4																										
5																										
6																										
7																										
8																										
9																										

More Info:

Check the CEMC at Home webpage on Thursday, April 16 for the solution to this problem. Alternatively, subscribe to Problem of the Week at the link below and have the solution, along with a new problem, emailed to you on Thursday, April 16.

This CEMC at Home resource is the current grade 3/4 problem from Problem of the Week (POTW). This problem was developed for students in grades 3 and 4, but is also appropriate for students in grades 5 and 6. POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students. Each week, problems from various areas of mathematics are posted on our website and e-mailed to our subscribers. Solutions to the problems are e-mailed one week later, along with a new problem. POTW is available in 5 levels: A (grade 3/4), B (grade 5/6), C (grade 7/8), D (grade 9/10), and E (grade 11/12).

To subscribe to Problem of the Week, to view this week's grade 5/6 problem, and to find many more past problems and their solutions, visit the Problem of the Week webpage.

Problem of the Week

Problem A and Solution

Mystery Code

Problem

James wants to send an image to a friend. The image is made up of filled in black squares on a grid. The rows of the grid are labelled with numbers from 1 to 9 and the columns of the grid are labelled from A to Z. James encodes the image by providing a series of short codes. Each short code consists of four parts:

- A letter between A and Z followed by a number between 1 and 9 indicating the column and row of the first square to be filled in on the grid,
- followed by an arrow ($\uparrow, \downarrow, \leftarrow, \rightarrow$) indicating the direction to fill,
- followed by a number, indicating the total number of squares to colour.

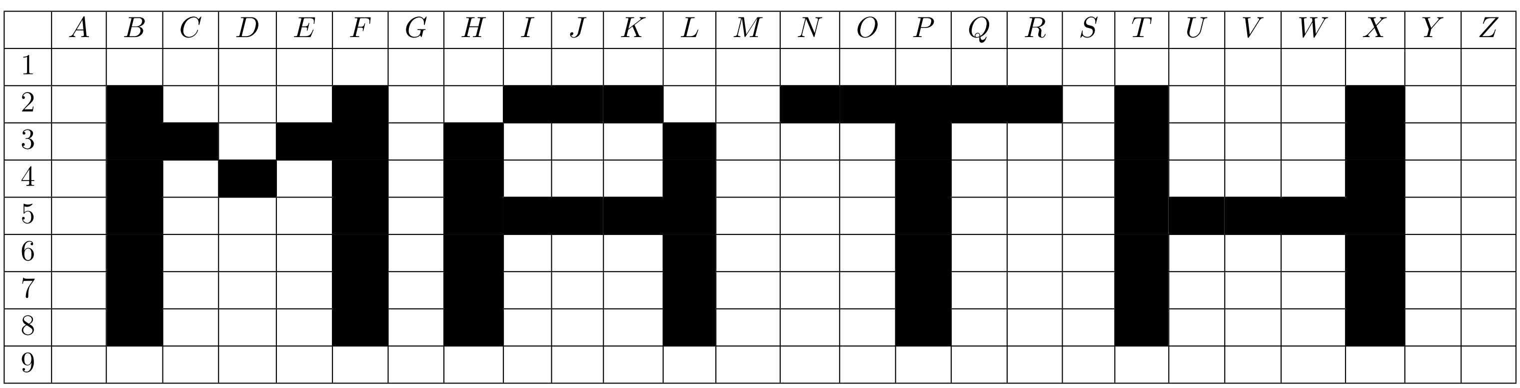
Determine the image that James sent using the following codes:

$$T8 \uparrow 7 \qquad D4 \leftarrow 1 \qquad R2 \leftarrow 5 \qquad I5 \rightarrow 3 \qquad F2 \downarrow 7 \qquad L8 \uparrow 6 \qquad C3 \rightarrow 1$$

$$P3 \downarrow 6 \qquad X2 \downarrow 7 \qquad E3 \uparrow 1 \qquad W5 \leftarrow 3 \qquad H8 \uparrow 6 \qquad K2 \leftarrow 3 \qquad B2 \downarrow 7$$

Solution

Here is the image:



Teacher's Notes

Digital images are formed by many individual small squares called *pixels*. With images like pictures taken on a phone, the pixels are usually different shades of colour. Before email attachments and scanning became so popular, businesses used fax machines to send scanned documents. A fax was a document scanned by a source machine and then transmitted as data over phone lines to another machine at the receiving end. The transmitted image was normally black and white, like the image in this problem.

The image in this problem is formed by 63 black pixels and 171 white pixels, for a total of $9 \times 26 = 234$ pixels. We could have described the image by identifying each individual pixel as being either black or white. This would require 234 bits of information. In this problem, we used 14 codes to represent the same information. Although the codes are more complicated to understand, overall they use less data to represent the same information. This concept of using less data to represent information is known as *compression*.

This particular method of compressing data is similar to a technique known as *run-length encoding*. This technique was used by fax machines to reduce the amount of data required to transmit an image over phone lines. Although fax machines are not very popular these days, run-length encoding is still used in other applications, including managing very long gene sequences in DNA research. There are many different techniques for compressing data. You are probably familiar with the results of many of them such as **jpg**, **zip**, and **mp3** files.