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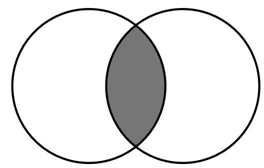
October 2017 Solutions

In honour of the 50th anniversary of the Faculty of Mathematics, at the beginning of each month of 2017, a set of five problems from the 54 years of CEMC contests will be posted. Solutions to the problems will be posted at the beginning of the next month. Hopefully, these problems will intrigue and inspire your mathematical mind. For more problem solving resources, please visit `cemc.uwaterloo.ca`.

1. *2010 Gauss Contest, Grade 8, Question 24*

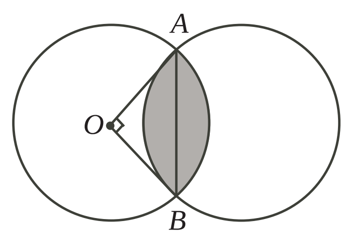
   Two circles each have radius 10 cm. They overlap so that each contains exactly 25% of the other's circumference, as shown. The area of the shaded region is closest to



   (A) 57.08 cm^2 **(B)** 55.24 cm^2 **(C)** 51.83 cm^2
   (D) 54.17 cm^2 **(E)** 53.21 cm^2

   Solution

   Label points A and B, the points of intersection of the two circles, and point O, the centre of the left circle.



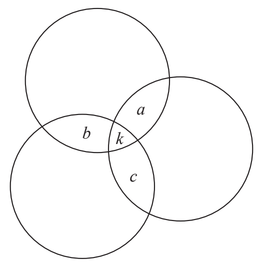
   Construct line segment AB, which by symmetry divides the shaded area in half.
   Construct radii OA and OB with $OA = OB = 10$ cm.
   Since each circle contains 25% or $\frac{1}{4}$ of the other circle's circumference, $\angle AOB = \frac{1}{4} \times 360^\circ = 90^\circ$.
   Thus, the area of sector AOB is $\frac{1}{4}$ of the area of the entire circle, or $\frac{1}{4}\pi r^2 = \frac{1}{4}\pi 10^2 = 25\pi$ cm^2.
   The area of $\triangle AOB$ is $\dfrac{OA \times OB}{2} = \dfrac{10 \times 10}{2} = 50$ cm^2.
   The area remaining after $\triangle AOB$ is subtracted from sector AOB is equal to half of the shaded area. Thus, the shaded area is $2 \times (25\pi - 50) \approx 2 \times (28.5398) = 57.0796$ cm^2.
   The area of the shaded region is closest to 57.08 cm^2.

ANSWER: (A)

2. *1998 Fermat Contest, Question 23*
Three rugs have a combined area of 200 m^2. By overlapping the rugs to cover a floor area of 140 m^2, the area which is covered by exactly two layers of rug is 24 m^2. What area of floor is covered by three layers of rug?

(A) 12 m^2 **(B)** 18 m^2 **(C)** 24 m^2 **(D)** 36 m^2 **(E)** 42 m^2

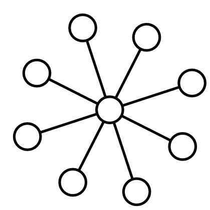
Solution
Draw the rugs in the following manner, where $a + b + c$ represents the amount of floor covered by exactly two rugs and k represents the amount of floor covered by exactly three rugs.



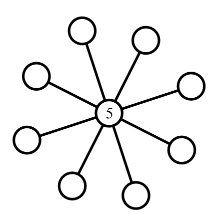
We are told that $a + b + c = 24$ m^2.
Since the total amount of floor covered when the rugs do not overlap is 200 m^2 and the total covered when they do overlap is 140 m^2, then 60 m^2 of rug is "wasted" on double or triple layers.
Thus, $a + b + c + 2k = 60$ m^2, since there are three layers of rug counted under the label k, and so $2k$ of "wasted" area.
Therefore, $(a + b + c + 2k) - (a + b + c) = 60$ $m^2 - 24$ m^2 and so $2k = 36$ m^2 which gives $k = 18$ m^2.
Thus, the area of floor covered by exactly three layers of rug is 18 m^2.

ANSWER: (B)

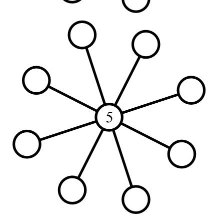
3. *2004 Galois Contest, Question 3*
In "The Sun Game", two players take turns placing discs numbered 1 to 9 in the circles on the board. Each number may only be used once. The object of the game is to be the first to place a disc so that the sum of 3 numbers along a line through the centre circle is 15.



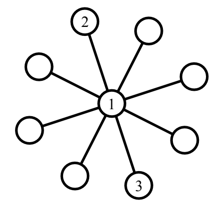
(a) If Avril places a 5 in the centre circle and then Bob places a 3, explain how Avril can win on her next turn.



(b) If Avril starts by placing a 5 in the centre circle, show that whatever Bob does on his first turn, Avril can always win on her next turn.



(c) If the game is in the position shown and Bob goes next, show that however Bob plays, Avril can win this game.



Solution

(a) If Bob places a 3, then the total of the two numbers so far is 8, so Avril should place a 7 to bring the total up to 15.
Since Bob can place a 3 in any the eight empty circles, Avril should place a 7 in the circle directly opposite the one in which Bob places the 3. This allows Avril to win on her next turn.

(b) As in (a), Bob can place any of the numbers 1, 2, 3, 4, 6, 7, 8, 9 in any of the eight empty circles. On her next turn, Avril should place a disc in the circle directly opposite the one in which Bob put his number. What number should Avril use? Avril should place the number that brings the total up to 15, as shown below:

Bob's First Turn	Total So Far	Avril's Second Turn
1	6	9
2	7	8
3	8	7
4	9	6
6	11	4
7	12	3
8	13	2
9	14	1

Since each of these possibilities is available to Avril on her second turn (since 5 is not in the list and none is equal to Bob's number), then she can always win on her second turn.

(c) Bob can place any of the numbers 4, 5, 6, 7, 8, 9 in any of the six empty circles. Which of these numbers can we pair up so that the sum of the pair is 14 (so that placing these numbers in the two circles at opposite ends of a line gives 15)? We can pair 5 and 9, and 6 and 8 in this way. However, the 4 and 7 do not pair up with any other number to give a sum of 14.
So if Bob on his next turn places 5, 6, 8, or 9, Avril should place the second number from the pair (that is, 9, 8, 6, or 5, respectively) opposite the number Bob places and she will win.
If Bob places the 4 or the 7, Avril should then place the 7 or the 4, respectively, in the opposite circle. She will not win on this turn, but this forces Bob to then place one of the remaining 4 paired numbers on his next turn, so Avril can win for sure on her following turn.

4. *2011 Canadian Intermediate Mathematics Contest, Question A6*
The product of all of the positive integer divisors of 6^{16} equals 6^k for some integer k. Determine the value of k.

Solution
We rewrite 6^{16} as $(2 \cdot 3)^{16}$, which equals $2^{16}3^{16}$.
If d is a positive integer divisor of 6^{16}, it cannot have more than 16 factors of 2 or of 3, and cannot include any other prime factors. Therefore, if d is a positive integer divisor of 6^{16}, then $d = 2^m 3^n$ for some integers m and n with $0 \leq m \leq 16$ and $0 \leq n \leq 16$.
Since there are 17 possible values for each of m and n, then there are $17 \times 17 = 289$ possible positive integer divisors d.

Since 6^{16} is a perfect square ($6^8 \times 6^8$), then every positive integer divisor other than its square root (6^8) can be paired with a different positive integer such that the product of the pair is 6^{16}. (For example, $2^4 3^{13}$ can be paired with $2^{12}3^3$ since $2^4 3^{13} \times 2^{12}3^3 = 2^{4+12}3^{13+3} = 2^{16}3^{16}$.)
Since there are 289 positive integer divisors in total, then there are $\frac{1}{2}(289-1) = 144$ of these divisor pairs whose product is 6^{16}.
To multiply all of the positive integer divisors together, we can multiply the product of the 144 pairs with the remaining divisor (6^8).
Therefore, the product of all of the positive integer divisors is $(6^{16})^{144} \cdot 6^8 = 6^{16(144)+8} = 6^{2312}$.
Thus, $k = 2312$.

5. *2017 Canadian Team Mathematics Contest, Relay #3*

(a) Suppose that $x = \sqrt{20 - 17 - 2 \times 0 - 1 + 7}$. What is the value of x?

(b) Let t be TNYWR.
If the graph of $y = 2\sqrt{2t}\sqrt{x} - 2t$ passes through the point (a, a), what is the value of a?

(c) Let t be TNYWR.
Suppose that

$$\frac{1}{2^{12}} + \frac{1}{2^{11}} + \frac{1}{2^{10}} + \cdots + \frac{1}{2^{t+1}} + \frac{1}{2^t} = \frac{n}{2^{12}}$$

(The sum on the left side consists of $13 - t$ terms.)
What is the value of n?

(In a Relay, the acronymn "TNYWR" in parts (b) and (c) stands for "the number you will receive", which is the answer to the previous part. Thus, the complete information necessary to solve problems (b) and (c) depends on the answers to problems (a) and (b), respectively.)

Solution

(a) Evaluating, $x = \sqrt{20 - 17 - 2 \times 0 - 1 + 7} = \sqrt{20 - 17 - 0 - 1 + 7} = \sqrt{9} = 3$.

(b) Since the graph of $y = 2\sqrt{2t}\sqrt{x} - 2t$ passes through the point (a, a), then $a = 2\sqrt{2t}\sqrt{a} - 2t$.
Rearranging, we obtain $a - 2\sqrt{2t}\sqrt{a} + 2t = 0$. We re-write as $(\sqrt{a})^2 - 2\sqrt{a}\sqrt{2t} + (\sqrt{2t})^2 = 0$ or $(\sqrt{a} - \sqrt{2t})^2 = 0$.
Therefore, $\sqrt{a} = \sqrt{2t}$ or $a = 2t$.
Since the answer to (a) is 3, then $t = 3$ and so $a = 6$.

(c) Multiplying both sides of the given equation by 2^{12}, we obtain

$$1 + 2^1 + 2^2 + \cdots + 2^{12-(t+1)} + 2^{12-t} = n$$

Since the answer to (b) is 6, then $t = 6$ and so we have

$$1 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 = n$$

Therefore, $n = 1 + 2 + 4 + 8 + 16 + 32 + 64 = 127$.