Warm-up Questions:

(a) Convert the angles with the following measures from degrees to radians: 180°, 90°, 60°, 45°, 30°, 48°.

\[
\begin{align*}
180° &= \pi, \\
90° &= \frac{\pi}{2}, \\
60° &= \frac{\pi}{3}, \\
45° &= \frac{\pi}{4}, \\
30° &= \frac{\pi}{6}, \\
48° &= \frac{4\pi}{15}.
\end{align*}
\]

(b) Convert the angles with the following measures from radians to degrees: \(\frac{\pi}{5}, \frac{5\pi}{6}, \frac{3\pi}{2}, \frac{7\pi}{4}\).

\[
\begin{align*}
\frac{\pi}{5} &= 36°, \\
\frac{5\pi}{6} &= 150°, \\
\frac{3\pi}{2} &= 270°, \\
\frac{7\pi}{4} &= 315°.
\end{align*}
\]

(c) Complete the chart below. The angles are given in radians.

<table>
<thead>
<tr>
<th>(\theta)</th>
<th>0</th>
<th>(\frac{\pi}{6})</th>
<th>(\frac{\pi}{4})</th>
<th>(\frac{\pi}{3})</th>
<th>(\frac{\pi}{2})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sin \theta)</td>
<td>0</td>
<td>(\frac{1}{2})</td>
<td>(\frac{\sqrt{2}}{2})</td>
<td>(\frac{\sqrt{3}}{2})</td>
<td>1</td>
</tr>
<tr>
<td>(\cos \theta)</td>
<td>1</td>
<td>(\frac{\sqrt{3}}{2})</td>
<td>(\frac{\sqrt{2}}{2})</td>
<td>(\frac{1}{2})</td>
<td>0</td>
</tr>
</tbody>
</table>

Question 1

Plot the points with Cartesian coordinates \(A(8\sqrt{3}, 8)\) and \(B\left(\frac{5}{4}, \frac{5\sqrt{3}}{4}\right)\) and then convert them to polar coordinates.

Solution: We first plot the point \(A(8\sqrt{3}, 8)\) in the plane.

\[
\text{Since } x = 8\sqrt{3} \text{ and } y = 8, \text{ we have}
\]
\[
r = \sqrt{x^2 + y^2} = \sqrt{(8\sqrt{3})^2 + 8^2} = \sqrt{192 + 64} = 16
\]

From the right-angled triangle in the diagram, we see we are looking for an angle \(\theta\) in the first quadrant that satisfies

\[
\sin \theta = \frac{y}{r} = \frac{8}{16} = \frac{1}{2}
\]

One possible choice is \(\theta = \frac{\pi}{6}\).

This means the point with Cartesian coordinates \((x, y) = (8\sqrt{3}, 8)\) can be described using polar coordinates \((r, \theta) = (16, \frac{\pi}{6})\).

Now we plot the point \(B\left(\frac{5}{4}, \frac{5\sqrt{3}}{4}\right)\) in the plane.
Since \( x = \frac{5}{4} \) and \( y = \frac{5\sqrt{3}}{4} \), we have
\[
x^2 + y^2 = \left( \frac{5}{4} \right)^2 + \left( \frac{5\sqrt{3}}{4} \right)^2 = \frac{25}{16} + \frac{75}{16} = \frac{100}{16} = \frac{25}{4}
\]
and so \( r = \sqrt{x^2 + y^2} = \frac{5}{2} \). From the right-angled triangle in the diagram, we see we are looking for an angle \( \theta \) in the first quadrant that satisfies
\[
\cos \theta = \frac{x}{r} = \frac{\left( \frac{5}{4} \right)}{\left( \frac{5}{2} \right)} = \frac{1}{2}
\]
One possible choice is \( \theta = \frac{\pi}{3} \).

This means the point with Cartesian coordinates \((x, y) = \left( \frac{5}{4}, \frac{5\sqrt{3}}{4} \right)\) can be described using polar coordinates \((r, \theta) = \left( \frac{5}{2}, \frac{\pi}{3} \right)\).

**Question 2**

Plot the points with Cartesian coordinates \( C(8, -8\sqrt{3}) \) and \( D(-\frac{5\sqrt{3}}{4}, -\frac{5}{4}) \) and then convert them to polar coordinates.

**Solution:** We first plot the point \( C(8, -8\sqrt{3}) \) in the plane.

Since \( x = 8 \) and \( y = -8\sqrt{3} \), we have
\[
r = \sqrt{x^2 + y^2} = \sqrt{8^2 + (-8\sqrt{3})^2} = \sqrt{64 + 192} = 16
\]
From the right-angled triangle in the diagram, we see we are looking for an angle \( \theta \) in the fourth quadrant for which the associated acute angle \( \alpha \) satisfies
\[
\cos \alpha = \frac{8}{16} = \frac{1}{2}
\]
This means \( \alpha = \frac{\pi}{3} \) and so one possible choice is \( \theta = \frac{5\pi}{3} \).

This means the point with Cartesian coordinates \((x, y) = (8, -8\sqrt{3})\) can be described using polar coordinates \((r, \theta) = (16, \frac{5\pi}{3})\).

**Note:** We could have instead observed that point \( C \) is related to point \( A \). They are the same distance from the origin, and their angles are complementary.

Now we plot the point \( D\left( -\frac{5\sqrt{3}}{4}, -\frac{5}{4} \right) \) in the plane.
Since \( x = -\frac{5\sqrt{3}}{4} \) and \( y = -\frac{5}{4} \), we have

\[
x^2 + y^2 = \left( -\frac{5\sqrt{3}}{4} \right)^2 + \left( -\frac{5}{4} \right)^2 = \frac{75}{16} + \frac{25}{16} = \frac{25}{4}
\]

and so \( r = \sqrt{x^2 + y^2} = \frac{5}{2} \). From the right-angled triangle in the diagram, we see we are looking for an angle \( \theta \) in the third quadrant for which the associated acute angle \( \alpha \) satisfies

\[
\sin \alpha = \left( -\frac{5}{2} \right) = \frac{1}{2}
\]

This means \( \alpha = \frac{\pi}{6} \) and one possible choice is \( \theta = \frac{7\pi}{6} \).

This means the point with Cartesian coordinates \((x, y) = \left( -\frac{5\sqrt{3}}{4}, -\frac{5}{4} \right)\) can be described using polar coordinates \((r, \theta) = \left( \frac{5}{2}, \frac{7\pi}{6} \right)\).

**Question 3**

Plot the point with Polar coordinates \( P(-1, \frac{11\pi}{6}) \) and then convert it to Cartesian coordinates.

**Solution:** First we represent \( P(-1, \frac{11\pi}{6}) \) with a positive value of \( r \). In other words, it is the point \( P(r, \theta) = P(1, \frac{5\pi}{6}) \). This is because this point lies on the line passing through the origin and making an angle of \( \frac{5\pi}{6} \). Also, the negative sign of \( r \) means that we move in the direction opposite to the direction defined by \( \theta = \frac{11\pi}{6} \).

Using the expressions of \( x \) and \( y \) in terms of \( r \) and \( \theta \), we see that

\[
x = \cos \left( \frac{5\pi}{6} \right) = -\frac{\sqrt{3}}{2}
\]

\[
y = \sin \left( \frac{5\pi}{6} \right) = \frac{1}{2}
\]

Therefore, the point has Cartesian coordinates \((x, y) = (-\frac{\sqrt{3}}{2}, \frac{1}{2})\).

**Activity Answers:**

<table>
<thead>
<tr>
<th>G</th>
<th>R</th>
<th>A</th>
<th>P</th>
<th>H</th>
<th>P</th>
<th>O</th>
<th>L</th>
<th>A</th>
<th>R</th>
<th>C</th>
<th>U</th>
<th>R</th>
<th>V</th>
<th>E</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>8</td>
<td>7</td>
<td>11</td>
<td>9</td>
<td>11</td>
<td>4</td>
<td>2</td>
<td>7</td>
<td>8</td>
<td>6</td>
<td>3</td>
<td>8</td>
<td>10</td>
<td>1</td>
<td>5</td>
</tr>
</tbody>
</table>

We now provide an explanation of each matching.

1. This point has polar coordinates \((4, 0)\). (E)

   Since \( r = 4 \) and \( \theta = 0 \), this is a point that is 4 units from the origin and lies on the ray defined by \( \theta = 0 \) which is the positive x-axis. This describes only point \( E \).
2. This point has polar coordinates \((4, \frac{3\pi}{2})\). (L)

This is a point that is 4 units from the origin and lies on the ray defined by \(\theta = \frac{3\pi}{2}\) which is the negative y-axis. This describes only point \(L\).

3. This point has polar coordinates \((4, \frac{3\pi}{4})\). (U)

This is a point that is 4 units from the origin lies on the ray defined by \(\theta = \frac{3\pi}{4}\). This describes only point \(U\).

4. This point could also be described using polar coordinates \((2, \frac{11\pi}{4})\). (O)

Note that \(\frac{11\pi}{4}\) and \(\frac{11\pi}{4} - 2\pi = \frac{3\pi}{4}\) are equivalent angles. So we are looking for the point with polar coordinates \((2, \frac{3\pi}{4})\). This is on the same ray as \(U\) above, but 2 units from the origin. This describes only point \(O\).

5. This point’s first coordinate, \(r\), satisfies \(r^2 = 2\). (S)

This means \(r = \pm\sqrt{2} \approx \pm 1.4\). It looks like the only point that is around 1.4 units from the origin is \(S\). You can draw a circle of radius 1.4 on the graph to confirm. This is describing point \(S\).

6. This point has the largest first coordinate, \(r\), out of all of the points. (C)

The point with the largest first coordinate will be the farthest from the origin. The point \(C\) is 5 units away and every other point appears to be closer than that. You can draw a circle of radius 5 on the graph to confirm! This is describing point \(C\).

7. This point has the smallest positive second coordinate, \(\theta\), out of all of the points. (A)

The point with the smallest positive second coordinate will make the smallest angle with the positive x-axis. This describes the point \(A\).

8. This point’s second coordinate, \(\theta\), satisfies \(2\sin \theta = 1\). (R)

If \(0 \leq \theta < 2\pi\) and \(\sin \theta = \frac{1}{2}\), then \(\theta = \frac{\pi}{6}\) or \(\theta = \frac{5\pi}{6}\). The only point that lies on the ray defined by \(\theta = \frac{\pi}{6}\) is \(R\) and there are no points that lie on the ray defined by \(\theta = \frac{5\pi}{6}\). This describes \(R\).

9. This point’s second coordinate, \(\theta\), satisfies \(\cos \theta = -1\). (H)

If \(0 \leq \theta < 2\pi\) and \(\cos \theta = -1\), then \(\theta = \pi\). The only point that lies on the ray defined by \(\theta = \pi\) (the negative x-axis) is \(H\).

10. This point’s first coordinate, \(r\), satisfies \(r = 3\). (V)

The only point that appears to be 3 units from the origin is \(V\). You can draw a circle of radius 3 on the graph to confirm. This is describing point \(V\).

11. This point’s coordinates satisfy \(r = \sin \theta\). (P)

Since \(-1 \leq \sin \theta \leq 1\), any coordinates that satisfy this equality must have \(-1 \leq r \leq 1\). The only point within 1 unit of the origin is \(P\). In fact, \(P\) appears to have polar coordinates \(r = 1\) and \(\theta = \frac{\pi}{2}\) which do satisfy \(\sin \theta = \sin \left(\frac{\pi}{2}\right) = 1 = r\).

12. This point’s coordinates satisfy \(r = \theta\). (G)

We are now left with one property (12) and one point \(G\). This means \(G\) must be the point satisfying \(r = \theta\). Using the distance formula, you can check that \(G\) is around 4 units from the origin. The ray through \(G\) is near the ray defined by \(\theta = \frac{5\pi}{4} \approx 4\), which provides some evidence that \(r \approx \theta\). (The actual point plotted has \(r = \theta = 4.1\).)