Grade 7/8 Math Circles
November 25, 2020
Introduction to Interest

Introduction

Interest can be seen as a reward for lending your money to a bank or other business. It can also be seen as a price for borrowing money. Suppose I take a loan from a bank. When I go to pay this money back, I’ll have to pay the amount I originally borrowed, plus a fee that is calculated as the cost of borrowing the money. For example, if I borrowed $1000 with a $100 cost to borrow, I will have to pay the bank back a total of $1100. This $100 is called interest, and is usually calculated as a rate.

Interest

Interest rates are expressed as percentages and must be converted into decimals in order to be used in equations. To convert a percentage into a decimal, divide the percentage by 100. This can be simply done using a calculator, or by moving the decimal two places to the left.

\[
75\% \rightarrow 75.0 \rightarrow 0.75
\]

Example: Convert the following percentages into decimals: 53\%, 8\%. 27.6\%, and 0.6\%.

Solution:

1. 53\% \quad 53 \div 100 = 0.53 \quad 53\% \rightarrow 53.0 \rightarrow 0.53
2. 8\% \quad 8 \div 100 = 0.08 \quad 8\% \rightarrow 08.0 \rightarrow 0.08
3. 27.6\% \quad 27.6 \div 100 = 0.276 \quad 27.6\% \rightarrow 27.6 \rightarrow 0.276
4. 0.6\% \quad 0.6 \div 100 = 0.006 \quad 0.6\% \rightarrow 00.6 \rightarrow 0.006

To calculate a percentage of a number, multiply the number by the decimal form of the percentage.
Example: Calculate 25% of 700.

Solution:

\[ 0.25 \times 700 = 175 \]

So 25% of 700 is 175.

Let’s take a look at some examples on calculating interest using different rates over specific periods of time.

**Simple Interest**

Under **simple interest**, the interest earned (or paid) is calculated by multiplying the original amount invested (or borrowed), called the **principal**, by the annual (yearly) interest rate. This is then multiplied by the length of the investment (or loan) in years.

The formula to calculate simple interest, \( I \), is:

\[ I = Prt \]

where \( P \) is the **principal** of the loan (or investment), \( r \) is the interest **rate**, and \( t \) is the **time** in years.

**Note:** \( Prt \) is equivalent to \( P \times r \times t \).

**Example:** What is the simple interest due on a $2500 loan at the end of 10 months if the annual interest rate is 7.5%?

**Solution:** To begin, it is a good idea to list the information that we need and the information that we are given. In this case, we want to calculate simple interest, \( I \), and we are given the principal \( P \), rate \( r \), and time \( t \). In particular,

\[ I = ? \]
\[ P = 2500 \]
\[ r = 7.5\% \]
\[ t = 10 \text{ months} \]

First of all, we need to convert our rate into a decimal.

\[ 7.5\% = 0.075 \]
Recall simple interest is based on time in **years**. This means we cannot simply plug 10 into our equation since it is months. We need to convert months into years. In this case,

10 months is equivalent to \( \frac{10}{12} \) of a year

Now we can substitute the information we have into our simple interest equation.

\[
I = Prt \\
= (2500)(0.075) \left( \frac{10}{12} \right) \\
= 156.25
\]

Thus the total interest due at the end of 10 months is $156.25.

**Accumulated value**, denoted by \( S \), is the **total** amount of a loan or investment after \( t \) years. Under simple interest, the accumulated value is the sum of the principle amount and the interest amount.

\[
S = P + I
\]

Since \( I = Prt \), this can be written as:

\[
S = P + Prt = P(1 + rt)
\]

**Note:** \( P(1 + rt) \) is equivalent to \( P \times (1 + rt) \)

**Aside:** How is \( P + Prt = P(1 + rt) \)? Let’s take a further look: [https://youtu.be/RiThQP7MU0](https://youtu.be/RiThQP7MU0)

Under simple interest, the formula to calculate the accumulated value, \( S \), is:

\[
S = P(1 + rt)
\]

where \( P \) is the principal of the loan (or investment), \( r \) is the interest rate, and \( t \) is the time in years.

**Example:** Suppose I took a $5000 loan from a bank which earns 6% simple interest annually. How much money will I owe if I pay back the loan in ten years? How much is the total interest I owe in 10 years?
What if you are asked to solve a problem where you are given the accumulated value, but not the principal amount? This is called a present value calculation. To solve this we can rearrange our simple interest equation to solve for $P$ instead of $S$. As a matter of fact, you can use your equation solving skills to solve for any unknown.

**Example:** How much would I have to deposit into my savings account today in order to have $1600 in 3 years if the account earns 5% simple interest annually?

Before we look at another type of interest, let’s take a look at a method of calculation that we will need for our next formula.

**Aside: Exponentiation**

Exponentiation is *repeated multiplication*.

An exponent and base looks like the following:

\[ a^3 \]

The small number written above and to the right of the number is called the **exponent**. The number underneath the exponent is called the **base**. In the example, \( 4^3 \), the exponent is 3 and the base is 4. We say “4 is raised to the **third power**”.

An exponent tells us to multiply the base by itself that number of times. In the example above, \( 4^3 = 4 \times 4 \times 4 \). Once we write out the multiplication problem, we can easily evaluate the expression. Let’s do this for the example we’ve been working with:

\[ 4^5 = 4 \times 4 \times 4 \times 4 \times 4 = 16 \times 4 = 64 \]

The main reason we use exponents is because it’s a shorter way to write out big numbers. For example, we can express the following long expression: \( 2 \times 2 \times 2 \times 2 \times 2 \times 2 \) as \( 2^6 \) since 2 is being multiplied by itself 6 times. We say 2 is raised to the **6th power**.

**Exercise:** Evaluate.

\[ 2^5 = \quad 1^9 = \quad (1 + 3)^3 = \quad (4 + 4)^2 = \]

**Video Solution:** [https://youtu.be/78iV0aCFGrQ](https://youtu.be/78iV0aCFGrQ)
Compound Interest

Under compound interest, the interest earned (or paid) is calculated on the original amount invested (or borrowed) and the interest already earned (or paid), during the length of the loan (or investment) in years.

Example: Assume that you have deposited $1000 in a savings account that pays 10% interest compounded annually.

- After one year, the money earns $1000 \times 0.1 = $100.
  The new balance is $1100 = 1000 \times (1 + 0.1)$.

- After two years, the money earns $1100 \times 0.1 = $110.
  The new balance is $1210 = 1100 \times (1 + 0.1) = 1000 \times (1 + 0.1) \times (1 + 0.1) = 1000 \times (1 + 0.1)^2$.

- After three years, the money earns $1210 \times 0.1 = $121.
  The new balance is $1331 = 1210 \times (1 + 0.1) = 1000 \times (1 + 0.1) \times (1 + 0.1) \times (1 + 0.1) = 1000 \times (1 + 0.1)^3$.

And so on.

Under compound interest, the formula to calculate accumulated value, $S$, is:

$$S = P(1 + r)^t$$

where $P$ is the principal of the loan, $r$ is the interest rate, and $t$ is the time in years.

Example: Let’s go back to the example above. We saw that under compound interest, the balance in the savings account after 3 years would be $1331. What would the balance be in the account after 3 years if it was under 10% simple interest?

Solution: Recall that the formula to calculate accumulated value under simple interest is $S = P(1 + rt)$.

We have that the principal amount $P$, is $1000$. The rate of interest $r$, is $10\% = 0.10$, and time $t$, is 3 years. Then,

$$S = P(1 + rt) = 1000(1 + (0.10)(3)) = 1300$$
The balance in the savings account after 3 years under simple interest is $1300.

Which interest type would you rather want for your savings account, compound or simple? I would want compound interest! After 3 years under simple interest, I earned $300 in interest ($1000 + 300). Whereas after 3 years under compound interest, I earned $331 of interest ($1000 + 331). In this case, I earn more money if my interest is compounded annually!

Examples:

1. I deposit $5000 into a savings account which earns 6% interest compounded annually. How much money will be in the account in ten years?

2. How much do I have to deposit today in order to have $5000 in 5 years if interest is 8.5% compounded annually?

Video Solution: https://youtu.be/O2Lq6svbQjE

Nominal Rates

So far we have only used compound interest rates that are compounded annually. The term “compounded annually” means that the interest is calculated and applied once a year. More frequent periods are also available. For example, interest could be calculated twice a year, four times a year, every month, every week, every day, etc. Because of this, we need to make some small changes to our formula to calculate the accumulated value under compound interest:

Under compound interest, the formula to calculate accumulated value, \( S \), is:

\[
S = P \left(1 + \frac{r}{m}\right)^n
\]

where \( m \) is the number of compounding periods in a year, \( r \) is the nominal rate of interest compounded \( m \) times a year, and \( n \) is the number of compounding periods over the entire life of the loan.

In the context of a question, how do you know what \( m \) is?

Ask yourself: “How many compounding periods can fit in one year?”

- “Compounded annually” \( \Rightarrow m = 1 \)
• “Compounded semi-annually” ⇒ \( m = 2 \)

• “Compounded quarterly” ⇒ \( m = 4 \)

• “Compounded monthly” ⇒ \( m = 12 \)

• “Compounded weekly” ⇒ \( m = 52 \)

• “Compounded daily” ⇒ \( m = 365 \)

**Note:** If \( t \) represents the number of years (as we’ve seen before), then we know that since \( n \) is the number of compounding periods over the entire life of the loan, we have \( n = m \times t \).

For example, if \( t \) is equal to 5 years and interest is compounded monthly, then the total number of compounding periods over 5 years, would be \( n = 12 \times 5 = 60 \).

**Example:** I deposit $5000 into a savings account which earns 6% interest compounded monthly. How much money will be in the account in ten years?

**Video Solution:** [https://youtu.be/1931nkDd3J8](https://youtu.be/1931nkDd3J8)