Intermediate Math Circles
Wednesday March 27, 2019
Problem Set 2 — Solutions

1. Solution

(a) \( \vec{u} + \vec{v} + \vec{w} = [3, 7] + [0, 4] + [2, -5] = [5, 6] \)
(b) \( 3\vec{u} - 2\vec{v} = 3[3, 7] - 2[0, 4] = [9, 21] - [0, 8] = [9, 13] \)
(c) \(-2\vec{u} + \frac{1}{8}\vec{v} + 3\vec{w} = [-6, -14] + [0, \frac{1}{2}] + [6, -15] = [0, -\frac{59}{2}] \)

2. Solution

\[
\begin{align*}
\begin{array}{c}
a[1, 1] + b[1, 1] + c[1, 1] - a[1, 2] + b[1, 2] - a[-1, -1] + b[-1, -1] + c[-1, -1] \\
= a([1, 1] - [1, 2] + [1, 1]) + b([1, 1] + [1, 2] + [-1, -1]) + c([1, 1] + [-1, -1]) \\
= a[1, 0] + b[1, 2]
\end{array}
\end{align*}
\]

3. Solution

(a) \( a = 4, b = 4 \)

We have \( a = 4 \). Thus, \( 6 = a + b \implies 6 = 4 + b \). Therefore, \( b = 2 \).
(b) \( a = 2, b = 6 \)

We have \( 2 = a \). Thus, \( b = 3a \implies b = 3(2) = 6 \).

4. Solution A)

(a) \( |\vec{v}| = \sqrt{9 + 16} = 5 \)
(b) \( \vec{u} = \frac{1}{5}[4, 3] = [\frac{4}{5}, \frac{3}{5}] \)
(c) \( |\vec{u}| = \sqrt{\left(\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25} + \frac{9}{25}} = \sqrt{\frac{25}{25}} = 1 \)

B) \( \vec{u} \) has the same direction of \( \vec{v} \) but \( \frac{1}{5} \) the magnitude of \( \vec{v} \)

5. Solution

(a) \( |\vec{v}| = \sqrt{25 + 144} = 13; \ vec{u} = \frac{1}{13}[5, 12] = [\frac{5}{13}, \frac{12}{13}] \)
\[
|\vec{u}| = \sqrt{\left(\frac{5}{13}\right)^2 + \left(\frac{12}{13}\right)^2} = \sqrt{\frac{25}{169} + \frac{144}{169}} = \sqrt{\frac{169}{169}} = 1
\]

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(b) \( |\vec{v}| = \sqrt{16 + 49} = \sqrt{65} \); \( \vec{u} = \frac{1}{\sqrt{65}} [4, 7] = \left[ \frac{4}{\sqrt{65}}, \frac{7}{\sqrt{65}} \right] \)

\[ |\vec{u}| = \sqrt{\left( \frac{4}{\sqrt{65}} \right)^2 + \left( \frac{7}{\sqrt{65}} \right)^2} = \sqrt{\frac{16}{65} + \frac{49}{65}} = \sqrt{\frac{65}{65}} = 1 \]

(c) \( |\vec{v}| = \sqrt{a^2 + b^2} \); \( \vec{u} = \frac{1}{\sqrt{a^2+b^2}} [a, b] = \left[ \frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}} \right] \)

\[ |\vec{u}| = \sqrt{\left( \frac{a}{\sqrt{a^2+b^2}} \right)^2 + \left( \frac{b}{\sqrt{a^2+b^2}} \right)^2} = \sqrt{\frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2}} = \sqrt{\frac{a^2+b^2}{a^2+b^2}} = 1 \]

6. Solution

(a) \( t=-1, \) pt(0,-3); \( t = 0, \) pt(3,-1); \( t = 1, \) pt(6,1)

(b) \( t=-1, \) pt(3,5); \( t = 0, \) pt(4,3); \( t = 1, \) pt(5,1)

7. Solution A)

(a) \( t=-1, \) pt A(1,6); \( t = 0, \) pt B(3,5); \( t = 1, \) pt C(5,4)
(b) \( t=-1, \) pt D(9,2); \( t = 0, \) pt E(5,4); \( t = 1, \) pt F(1,6)
B) They are the same line.

8. **Solution**

   (a) The two direction vectors are [2,-1] and [-4,2]. Since [-4,2] = -2[2,-1], they are parallel to each other and have the 'same' direction.

   (b) Sub 3 into the x of second equation 3 = 5 - 4t \(\implies\) -2 = -4t \(\implies\) t = \(\frac{1}{2}\)

Check with y; y = 4 + 2\(\frac{1}{2}\) = 4 + 1 = 5 \(\implies\) the point is on the line.

Therefore the lines are the same.

9. **Solution**

The two direction vectors are [3,-2] and [6,-4]. Since [6,-4] = 2[3,-2] they have the 'same' direction.

Show [6,5] is in the other line.

Sub 6 into the x \(\implies\) 6 = 9 + 6t \(\implies\) -3 = 6t \(\implies\) t = \(-\frac{1}{2}\)

Check with y; y = -4 \(-\frac{1}{2}\) = 2 \(\implies\) the point is not on the line.

Therefore the lines are not the same.

10. **Solution**

Rewrite the second equation in slope y-int form. 3x + 2y - 19 = 0 \(\implies\) 2y = -3x + 19 \(\implies\) y = \(-\frac{3}{2}\)x + \(\frac{19}{2}\). Therefore the slope is \(-\frac{3}{2}\) which gives a direction vector of [-2,3]. Therefore the two lines have the same direction.

Now sub in the point from a) into b): 3(3) + 2(5) - 19 = 19 - 19 = 0. This is a true statement and therefore the point is on the second line. And they are the same line.