Games and the Mathematical Process, Week 1

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1 Class Problems

For every game we played, we asked the following questions:

1. Should you go first or second? What moves do you want to make?
2. Is there a strategy that will work every time? Can you win every time you play?
3. If there's a strategy that will work every time, why will it work every time?

Problem 1.1. Thai 21:

(a) We start with 21 flags. Two players play a game: On each player's turn, they can take away 1, 2, or 3 flags. The player who takes the last flag wins.

(b) What if the player who takes the last flag loses instead?

Problem 1.2. Half a Chessboard: Two players play the following game on a chessboard: Four dots are placed on an eight-row chessboard, one in each of the first four columns. (Dots may start in any row, and do not need to start on the same row.) On each player's turn, they must move one, two, three, or all four dots one square downward. The player who cannot make a move loses.

Problem 1.3. Pick and Split: There are two piles of stones. One pile has n stones, and the other pile has m stones. (n and m are whole numbers.) On each player's turn, they must discard one pile of stones and split the other pile into two piles. Every pile must have at least one stone. The first player unable to move loses.

Problem 1.4. Race to 1000: Two players play a game on a blackboard. The first player begins by writing the number 2. Then, on each player's turn, they first select a whole number smaller than the number written on the blackboard. Then, they replace the number on the board with the sum of the previous number and their selected number. The player who writes the number 1000 wins.

2 Problems to Ponder

Preparing for the Journey

Problem 2.1. Generalise Thai 21: What if we start with n flags, each player can take away any number of flags between 1 and k inclusive on their turn, and the player who takes the last flag wins? What if the player who takes the last flag loses instead?

Problem 2.2. What would the winning strategy for Race to 1000 be if the player who writes a number greater than or equal to 1000 loses instead?

Problem 2.3. Consider the following variants on Thai 21. What are their winning strategies?

(a) We start with 21 flags. On each player's turn, they can take away 1, 2, or 4 flags. The player who takes the last flag wins. What if the player who takes the last flag loses instead?

(b) We start with 82 flags. On each player's turn, they can take away 1, 4, or 5 flags. The player who takes the last flag wins. What if the player who takes the last flag loses instead?

(c) We start with n flags. On each player's turn, they can take away 1, 3, or 5 flags. The player who takes the last flag wins. What if the player who takes the last flag loses instead?

Problem 2.4. Two players play a game on a blackboard.

(a) The first player begins by writing the number 1. Then, on each player's turn, they multiply the number written on the board by any whole number between 2 and 9 inclusive and write the new number on the board. The first player to write a number larger than 1000 wins.

(b) What if the first player to write a number larger than 1000 loses instead?

Going to the Castle

Problem 2.5. What would the winning strategy for Half a Chessboard be if the player who moves last loses instead?

Problem 2.6. Barbara and Jenna play the following game, in which they take turns. A number of coins lie on a table. When it is Barbara's turn, she must remove 2 or 4 coins, unless only one coin remains, in which case she loses her turn. When it is Jenna's turn, she must remove 1 or 3 coins. A coin flip determines who goes first. Whoever removes the last coin wins the game. If both players use their best strategy, who will win the game if the game starts with 2018 coins?

Problem 2.7. Two players play a game.

(a) A checker is placed at each end of a strip of 20 squares. On each player's turn, they must move one of the checkers in the direction of the other by one or two squares. A checker cannot jump over another checker. The player who cannot move loses.

(b) What if the player who cannot move wins instead?

Problem 2.8. Two players play a game starting with n coins. On each player's turn, they may take away either one coin or any prime number of coins (that is, any of the numbers 2, 3, 5, 7, 11, ...). The player who takes the last coin wins.

Problem 2.9. Two players play a game starting with n flags. On each player's turn, they can take away any number of flags that is a power of 2. (The powers of 2 are the numbers 1, 2, 4, 8, 16, ...) The player who takes the last flag wins.

Problem 2.10. Two players play a game starting with n counters. On each player's turn, they can take away any number of counters which is a divisor of the current number of counters remaining, but all the counters can be taken only if there are 1 or 2 flags remaining. The player who takes the last counter wins.

Problem 2.11. Two players play a game starting with n stones. On each player's turn, they may take away any prime power of stones - that is, any number of stones that can be written in the form p^m, where p is a prime number and m is a nonnegative integer (any of the numbers 0, 1, 2, 3, ...). The player who takes the last stone wins.

Problem 2.12. Two players play a game starting with n candies. On each player's turn, they can take away no more than half the remaining candies, but they must take at least one candy. The player who takes the last candy wins.

Oh No, the Castle was Filled with Bees

Problem 2.13. Two players play a game starting with n stones. On the first player's turn, they may not take all of the stones. On each subsequent player's turn, they can take away at most twice the number of stones that the previous player removed (but must take away at least one stone). The player who takes the last stone loses.

Problem 2.14. Four heaps contain 38, 45, 61, and 70 matches respectively. Two players take turns choosing any two of the heaps and removing a non-zero number of matches from each heap (not necessarily the same number from each). The player who cannot make a move loses. Which one of the players has a winning strategy? Can you generalise?

Solutions to these problems will not be given. For hints or solutions to individual problems, feel free to email the presenter at `kris.siy@uwaterloo.ca`*. Problem sources are also available from the presenter by request.*