UNIVERSITY OF
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Mathematics and Computing

Grade 6 Math Circles

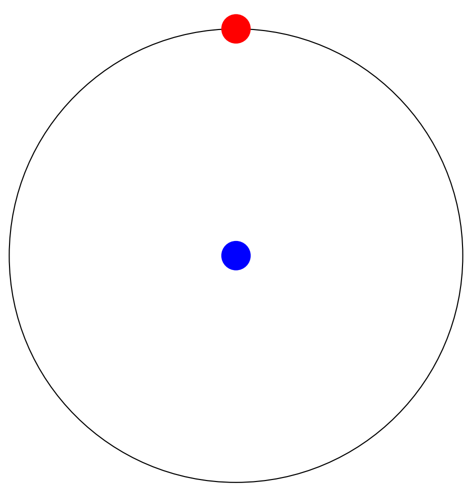
October 9 & 10 2018

Visual Vectors

Introduction

What is a **vector**? How does it differ from the numbers you already know and understand? Let's explore.

In the picture below, the outline of the circle is always 3 cm away from the blue dot in the middle (in other words, the *radius* of the circle is 3 cm). If you ask me to draw a red dot 3 cm from the blue one, that means I could draw it anywhere on this circle. What if you wanted me to draw the red dot in a particular place on the circle? How would you tell me to do that? Would I know where to draw it if you just said "draw a red dot 3 cm from the center of the blue dot"?



For me to know exactly where you want the red dot to go, you would have to tell me a direction from the blue dot to put it in. Telling me 3 cm above, or North, or 3 cm away at 90° from the horizontal, all tell me to exactly where to draw the dot.

Scalars vs. Vectors: the difference of direction

Telling me to go 3 cm from the blue dot is giving me a ***scalar***: it only tells you a size, or how much there is of something. In this case, the "how much" is a distance: 3 cm of distance. If you tell me you're moving at 20 km/h, you're still giving me a scalar: you're only telling me "how much" speed you have, and nothing else. All the numbers that you're used to seeing are scalars.

A ***vector***, on the other hand, tells you two things instead of one: it still tells you "how much", but it also tells you "in which direction". If you tell me to draw a red dot 3 cm from the blue dot, you're only giving me a scalar, without a direction. Telling me to draw the red dot 3 cm [up the page from the blue dot] is giving me a vector. With a vector, I also know which direction to go. The direction part of a vector usually comes after the number and its unit, inside square brakets like this: [ ]

As another example: telling me that you're moving at 20 km/h is a scalar. Telling me that you're moving at 20 km/h [North] is a vector. With the vector, you tell me how much speed you have, and which direction that *speed is going*.

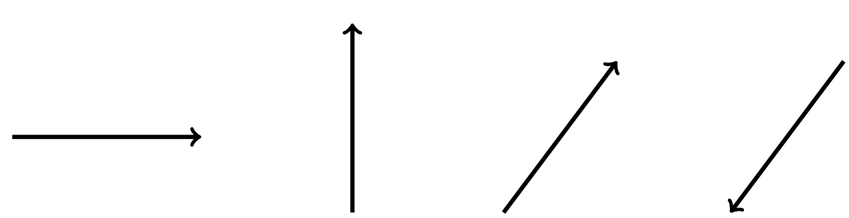
Drawing Vectors

Arrows also have a size and direction like vectors do. This is what we'll use to draw vectors:

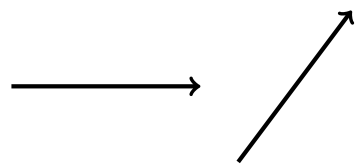
Size is represented by how long the arrow is.

Direction is represented by which way the arrow is pointing.

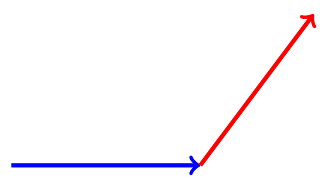
Practice: Measure these vectors with a ruler, and write out their size and which direction they are pointing in.



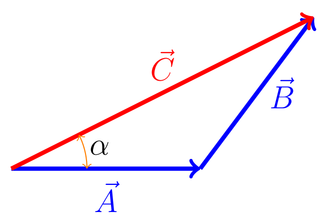
Vector Operations: addition



Adding vectors together is as simple as following the arrows. If I want to add these two vectors together, then I need to follow one, and then the other.



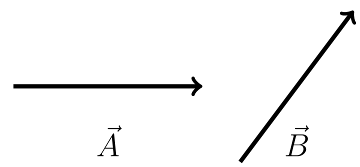
The result looks like this:



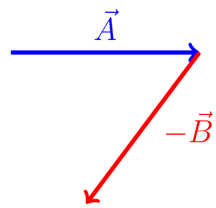
Note: We draw a little arrow on top of letters to know that they represent vectors

Practice: What is the final vector $\vec{C}$? Give your answer in centimeters, and use the angle α to give a direction. Your direction should say something like "______ degrees above/below A"

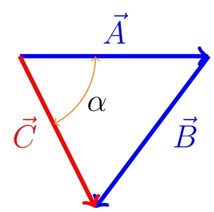
Vector Operations: subtraction



To subtract vectors $\vec{A} - \vec{B}$ we use a similar strategy. First, we follow vector $\vec{A}$, then go <u>backwards</u> along $\vec{B}$, by turning it around



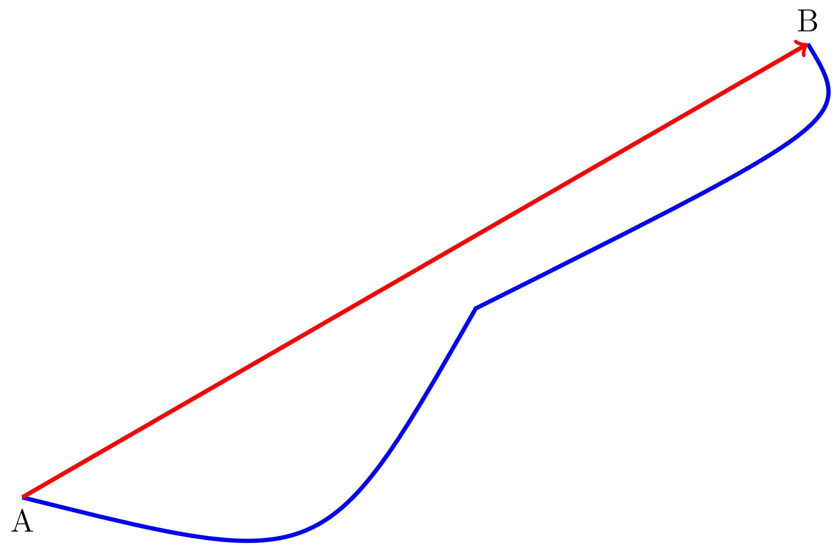
The result looks like this:



Note: We draw a little arrow on top of letters to know that they represent vectors

Practice: What is the final vector $\vec{C}$? Give your answer in centimeters, and use the angle α to give a direction. Your direction should say something like "______ degrees above/below A"

Application: distance vs displacement



Distance How long was the path you took?

Displacement How far are you from where you started?

Distance is the *scalar* that tells you "how much length there is". Displacement is the *vector* that tells you "how much length there is, and what direction it is in".

If you imagine walking from point A to point B in the diagram above, then the distance you've walked changes depending on the path you take. Walking along the blue path would take longer, and you'll have walked a greater distance.

No matter which path you take though, your displacement will NOT change. No matter what path you draw or decide to take, it doesn't change the fact that point B is "12 cm [30° up from the right]" from point A.

For distance and displacement to be the SAME, you would need to take the shortest, straightest path.

Exercise: Draw small displacement vector arrows along the blue path and add all the vectors together. Do you see that you will always get the red vector as your final displacement, no matter which path you take?

Problems

REVIEW

1. What is the difference between a scalar and a vector? What two things will a vector tell you? Give an example of each.

   Scalar: only tells you about a size, or "how much" there is of a unit. Ex. 5 m, 24 km/h, 6 mm are all scalars

   Vector: tells you two things: a size and a direction. 24 km/h [East] is a vector.

   The difference between them is that vectors have a direction, that you have to be careful of when doing operations like addition or subtraction.

2. For each of the following, state whether the quantity given is a scalar or a vector.

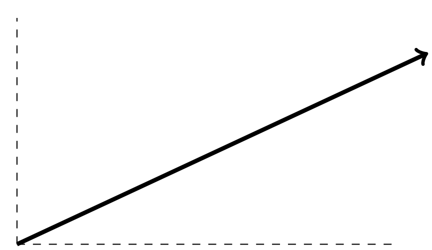
   (a) Jason moved his desk 5 m. Scalar

   (b) Cindy is driving her car at 50 km/ h [North].Vector

   (c) A ball is falling down at 3 m/s. Vector

   (d) Alice is hiking through a forest. She's going directly South-East.
   Neither, this is just a direction.

3. How would you describe the direction of the following vector on a page? How would you describe it in terms of compass directions? You do not need to measure the size of this vector, but do give an angle. (note: try to give two descriptions using compass directions)

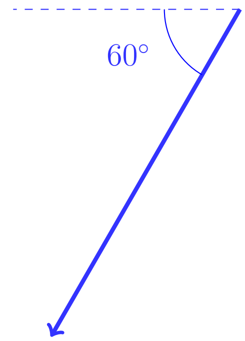


   This vector can be described as any of the following:
   On paper: 6 cm [25° above the right horizontal], or 6 cm [65° right of being vertical upwards]
   If seen as compass directions: 6 cm [25° North of East], or 6 cm [65° East of North]

APPLY

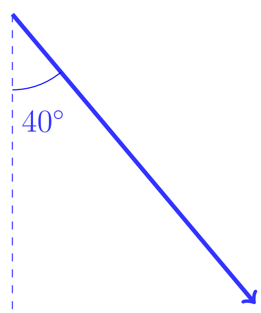
4. For each of the following, draw the vector being described.

   (a) 5 cm [60° below left]



   (b) A car is driving along the highway at 100 km/h [East]
   (note: you can draw it so that 1 cm = 10 km/h)

   (c) A boat is floating down a river at 50 km/h [40° East of South]

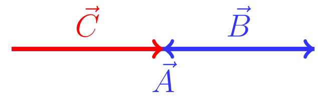


5. Add these vectors together. What is the resulting vector? Measure the size and write the direction.

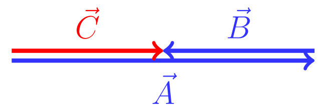
   (a) 

   To add the vectors together, we say that we'll "first follow one, then the other".

   For additions it doesn't matter which one you follow first, you'll still end up in the same place (just like with scalar additions). The two vectors are along the same line. The vector addition looks like this:



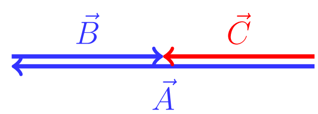
where the red vector $\vec{C}$ is the resultant vector. To make this more clear to see, we can draw the addition like this if we want:



These vectors are actually overlapping as they are in the first diagram, but can be drawn this way so we can clearly see them all.
Measuring the resultant, we get that $\vec{C} = 2$ cm [right]

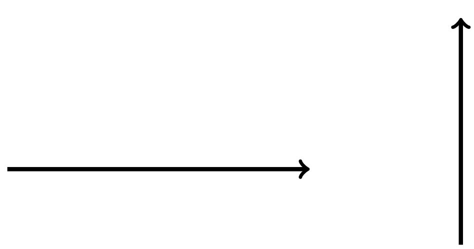
(b) 

As before, it doesn't matter which vector we follow first: $\vec{A} + \vec{B} = \vec{B} + \vec{A}$

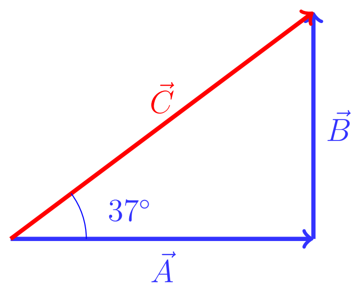


The resultant vector is $\vec{C} = 2$ cm [left]

6. Add these vectors together. What is the resulting vector? Measure the size and write the direction as [degrees above or below the left or right].

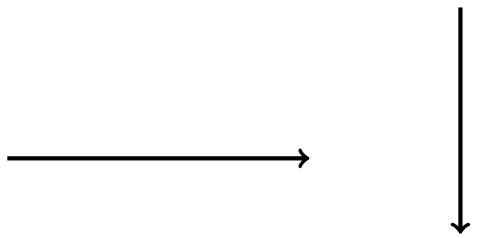
   (a) 

   We need to put the vectors together so we can add them together. After following one and then the other, the resultant vector goes from where you started (at the beginning of the first vector) to where you ended (at the end of the second vector).

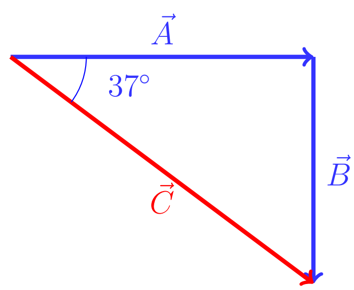


The resultant vector is $\vec{C} = 5$ cm [37° above the right]. It's important that when lining up the vectors, their size and direction is not changed: you're only allowed to move the **whole vector** to line up for addition.

(b)



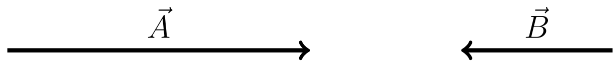
Follow the same process as for part (a) above.



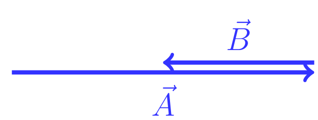
The resultant vector is $\vec{C} = 5$ cm [37° below the right].
As an exercise, try adding $\vec{B} + \vec{A}$ and compare your result.

7. Subtract these vectors. What is the resulting vector? Measure the size and write the direction.

(a) Calculate both i. $\vec{A} - \vec{B}$ and ii. $\vec{B} - \vec{A}$

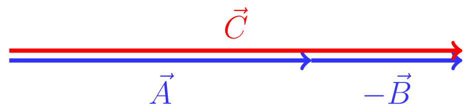


i. As always, we'll first put the vectors together:



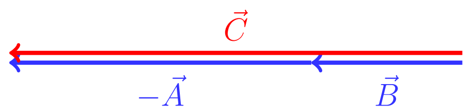
*note: as explaind in the solution of 5 (a), these vectors are actually overlapping, but can be drawn separated like this so it's clear to see that's happening. For $\vec{A} - \vec{B}$, vector $\vec{B}$ is the *subtractor*. So, we need to first follow $\vec{A}$ as it is, then *turn* $\vec{B}$ *around*, so that it still starts where A ends, but is facing the opposite way.

We can now follow $-\vec{B}$ to its end, and get the resultant vector going from where we started (at the start of $\vec{A}$) to where we've ended (end the end of $-\vec{B}$).



The resultant vector is $\vec{C} = 6$ cm [right]

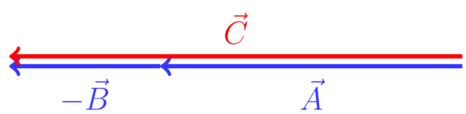
ii. Following the steps outlined in 7 (a) i. (right above), for $\vec{B} - \vec{A}$ we get the following:



The resultant vector is $\vec{C} = 6$ cm [left]

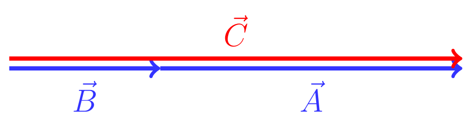
(b) Calculate both i. $\vec{A} - \vec{B}$ and ii. $\vec{B} - \vec{A}$

i. Following the steps outlined in 7 (a) i., for $\vec{A} - \vec{B}$ here we get the following:



The resultant vector is $\vec{C} = 6$ cm [left]

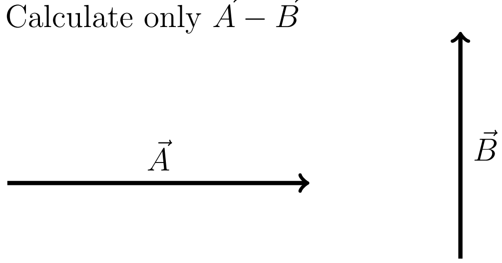
ii. Following the steps outlined in 7 (a) i., for $\vec{B} - \vec{A}$ here we get the following:



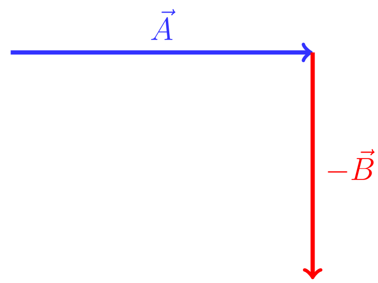
The resultant vector is $\vec{C} = 6$ cm [right]

8. Subtract these vectors. What is the resulting vector? Measure the size and write the direction as [degrees above or below the left or right].

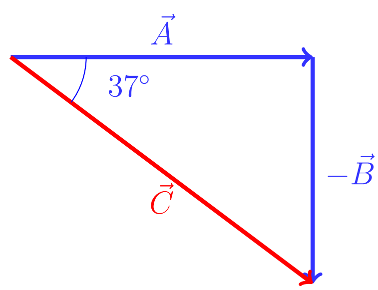
   (a) Calculate only $\vec{A} - \vec{B}$

   Following the steps outlined in 7 (a) i., we first line up the vectors as we would for an addition, then *turn around the subtractor* $\vec{B}$ so it's facing the opposite direction (and call this $-\vec{B}$)

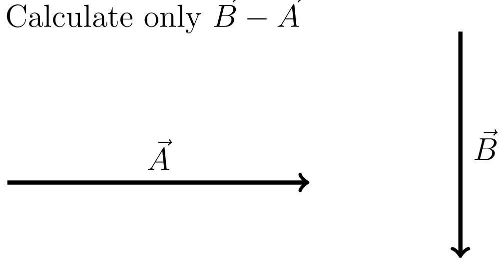
   

   The resultant vector goes from where we started (the the beginning of $\vec{A}$) to where we've ended off (at the end of $-\vec{B}$).

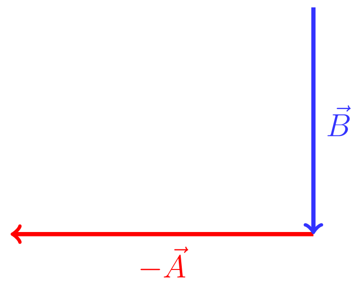
   

   The resultant vector is $\vec{C} = 5$ cm [37° below the right]

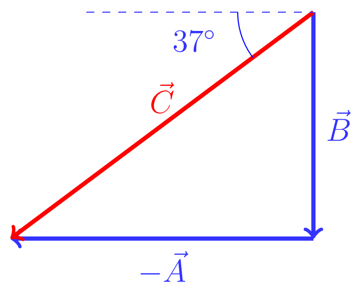
   (b) Calculate only $\vec{B} - \vec{A}$

   Following the steps outlined in 7 (a) i., we first line up the vectors as we would for an addition, then *turn around the subtractor* $\vec{A}$ so it's facing the <u>opposite</u> direction (and call this $-\vec{A}$)



The resultant vector goes from where we started (the the beginning of $\vec{B}$) to where we've ended off (at the end of $-\vec{A}$).



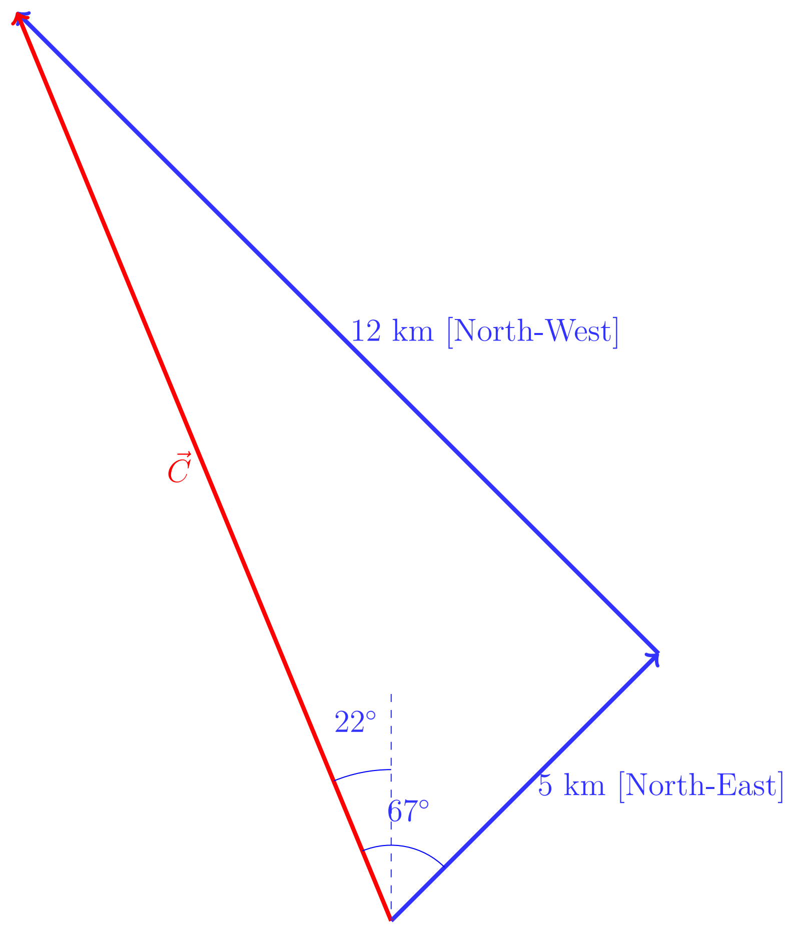
The resultant vector is $\vec{C} = 5$ cm [37° below the left]

9. Answer the following questions, doing the vector additions/subtractions you need.

   (a) Carl is out for a walk. He first goes 5 km [North-East], and then walks 12 km [North-West]. What is his final displacement? What is the distance he's travelled?
   The total distance Carl has travelled is found by simple scalar addition:

$$5\,\text{km} + 12\,\text{km} = 17\,\text{km}$$

   To find Carl's total displacement, we need to use vector addition. We let 1 cm in our diagram equal 1 km for Carl.

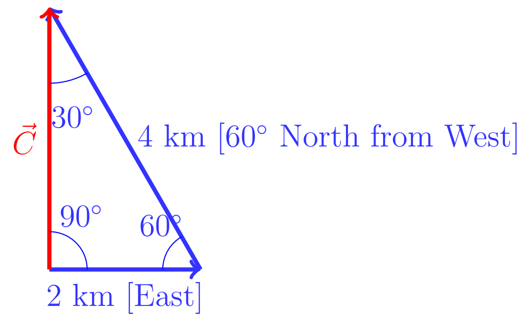


The resultant vector is $\vec{C}$ = 13 km [22° West of North] (rounded to the nearest degree). The angle of the $\vec{C}$ is 67° from the "5 km [North-East]" vector. We know that the direction North-East is 45° East of North, so we do $67° - 45° = 22°$ to get the angle of the $\vec{C}$ <u>from North</u>

(b) Bob is out for a walk. He first goes 2 km [East], and then walks 4 km [60° North from West]. What is his final displacement? What is the distance he's travelled? The total distance Bob has travelled is found by simple scalar addition:

$$2\,\text{km} + 4\,\text{km} = 6\,\text{km}$$

To find Bob's total displacement, we need to use vector addition. We let 1 cm in our diagram equal 1 km for Bob.

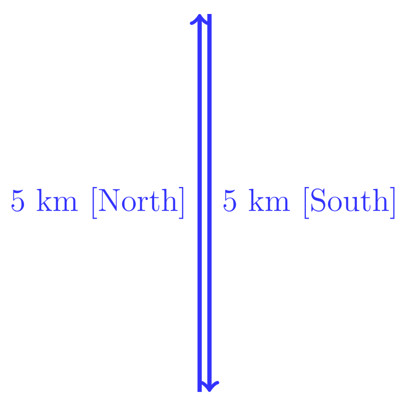


The resultant vector is $\vec{C}$ = 3.5 km [North] (distance rounded to one decimal place). The angle of the $\vec{C}$ is 90° from the "2 km [East]" vector, which we can measure with our protractor. (Notice also just for fun that the sum of the inside angles of a triangle is 180°)

(c) Alice goes to the super market 5 km North of her house and back. Assuming she took a straight path, what is the distance she travelled? What is her displacement? The total distance Alice has travelled is found by simple scalar addition:

$$5\,\text{km} + 5\,\text{km} = 10\,\text{km}$$

To find Alice's total displacement, we need to use vector addition. We let 1 cm in our diagram equal 1 km for Alice. Assuming she takes a completely straight path, she first goes 5 km [North], then comes back from the super market by going 5 km [South]



*note: These two vectors do actually overlap, but are drawn so that you can see them both clearly.

See that after following the arrows to do a vector addition, we end up in the same place as where we started (just like Alice did). This tells us that the answer is actually

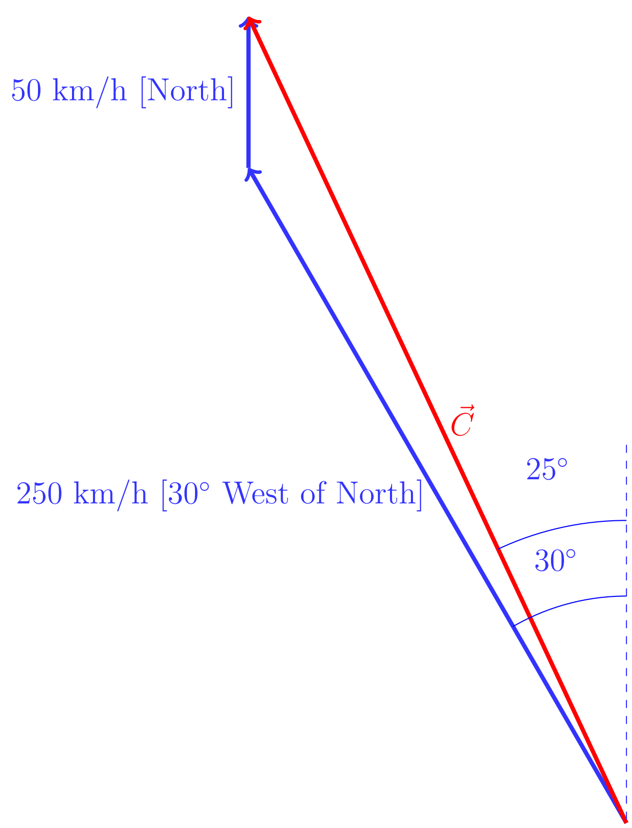
$$5\,\text{km[North]} + 5\,\text{km[South]} = \vec{0}$$

Notice that the answer is given as $\vec{0}$, and not just 0. When adding or subtracting vectors, the answer must always be a vector as well. The "zero vector" $\vec{0}$ is the only vector that <u>does not</u> have a size or direction.

CHALLENGE

10. In question 3 you drew vectors with units of *speed* (ex. 100 km/h [East]). These are call ***velocity*** vectors. The size of a velocity vector tells you the speed something is moving at, while it's direction tells you the direction it is going. Velocity vectors can be added and subtracted just like displacement vectors can, but will result in another velocity vector.

    (a) A plane is flying through the sky. Without any wind it would be flying at 250 km/h [30° West of North]. There is a wind pushing it as well, blowing 50 km/h [North]. What is the plane's final velocity?
    At any one point in time, the plane's final velocity will be the speed and direction it is trying to fly in, **plus** the speed and direction of the wind. This means we need to use vector addition. In the diagram below, let 1 cm equal 25 km/h for the plane.



The resultant vector is $\vec{C}$ = 294 km/h [25° West of North]

(b) Tom wants to row across a river. He can row at 5 km/h. The river has a current of 3 km/h [North]. If he wants to end up directly East across the river from where he started, what direction does he need to row in? What would his total velocity be?

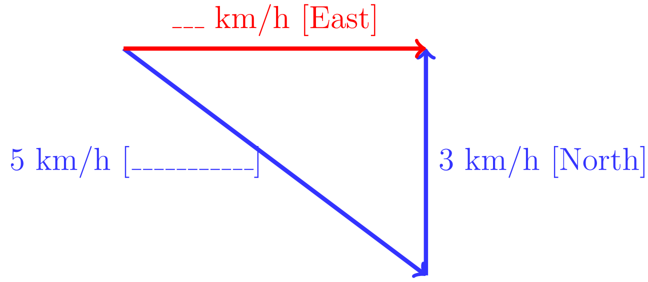
In this question, it is important to look at what we're given. We are given:

i. Tom's rowing speed (NOT his velocity).

ii. The river's velocity.

iii. The direction of his final velocity (NOT the speed)

To solve this problem, we can draw the diagram for the vector addition with what we know, then use that to figure out what we don't know.



In this diagram, 1 cm equals 1 km/h. We can now use a ruler and protractor to measure the things we don't know.

His total velocity would be 4 km/h [East]. The direction he needs to be rowing is [37° South of East]

(c) Tom is back at the river, but the wind is strong today. The river has a current of 12 km/h [North], so to get across, Tom decides to use a motor boat. He again wants to end up directly East across the river from where he started. If his total velocity going across the river is 5 km/h [East], what is the speed of the motor boat?

In this question, it is important to look at what we're given. We are given:

i. The river's velocity.

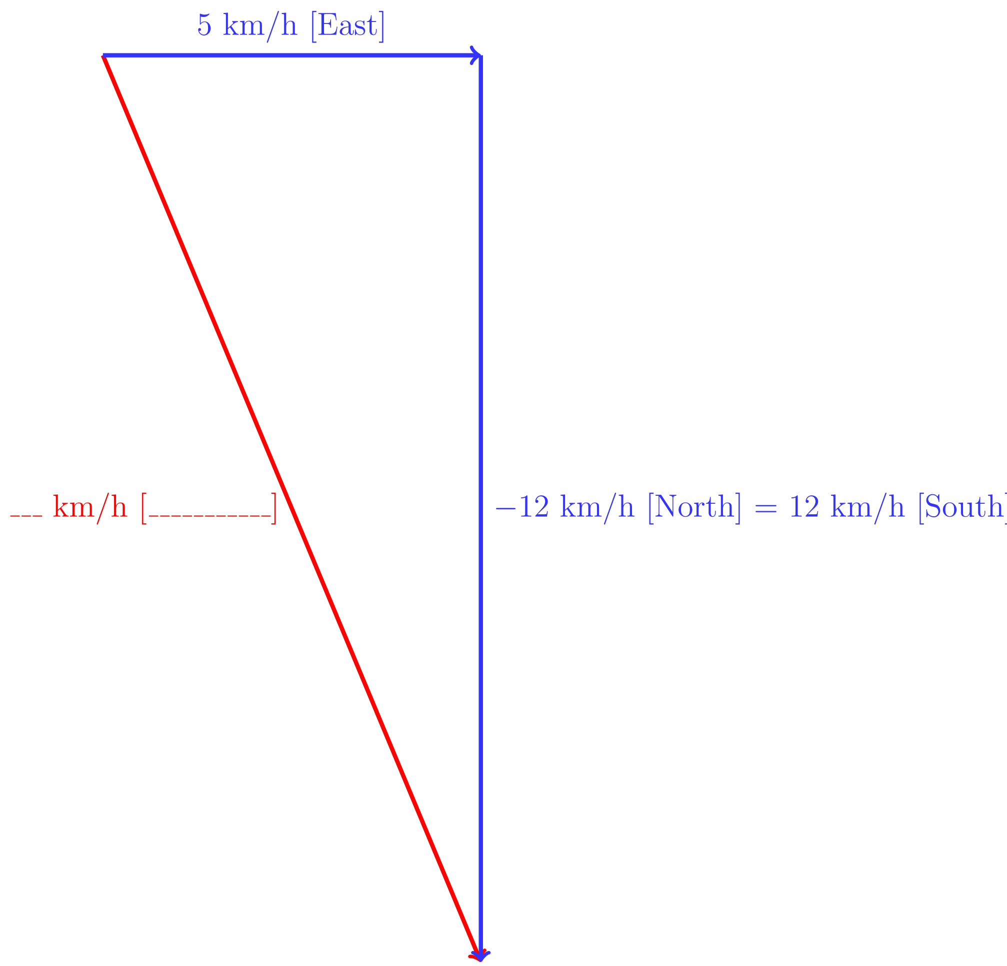
ii. Tom's final velocity

We know that

$$\textit{Velocity of the motor boat} + \textit{Velocity of the river} = \textit{Tom's final velocity}$$

. Just like with numbers, we can turn this into a subtraction to say

$$\textit{Tom's final velocity} - \textit{Velocity of the river} = \textit{Velocity of the motor boat}$$



In this diagram, 1 cm equals 1 km/h. We can now use a ruler to measure the motor boat's speed. The speed of the motor boat is 13 km/h.