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Centre for Education in
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Grade 6 Math Circles

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Structure of Math

Introduction

Where does math fit in the rest of the world? What makes something a math problem? What links addition, subtraction, and multiplication, to things like geometry and patterns? The answer is the way that we understand them. It's the structure of the problem that makes it a math problem. Math is not a specific problem, but a *way of doing problems*.

Math is everywhere in life, and it's amazingly useful, but it's not the only thing in the world. Math is never a universal tool. Math doesn't solve every problem. It fits somewhere in the world to be able to solve particular types of problems in particular ways. Everything else is also extremely important. Everything humans have continued to study solves some problem in some way. But of course, we're just talking about math today.

Logic and Proofs

We've said that math is really a *way of solving problems*. So what way is that? Math does almost everything with ***deductive logic***.

There are two main types of logic: *deductive* and *inductive*. Deductive logic is logic that is always certain about the outcomes it gives you. It is also called "formal logic". As great as it is that you can always be certain about your statements, deductive logic is actually pretty useless in a lot of everyday life. Inductive logic is the more common type of everyday logic. Inductive Logic is is also called "informal logic".

As an example, say you wake up in the morning and it's wet outside, with puddles up and down the street. In this case, you would reasonably assume that it was raining last night, and you're probably right. But you can't be certain. What if someone just spent the night going up and down the street with a hose making everything wet? That sounds ridiculous of course. But the reason that sounds ridiculous is because of *inductive logic*. Inductive logic decides on what is reasonable based on past experiences, but it can never be absolutely certain about its conclusions. To turn this example into a *deductive logic* question, we first need some "logical statements". So we can state this as:

1. if the ground outside is wet, then it rained outside
2. the ground outside is wet

From this we can say using deductive logic that, as long as the above statements are true, it definitely rained outside. Obviously the statements we just made are not always true. But deductive logic doesn't care so much about if everything is true. The point instead is that as long as the beginning statements are true, we know for sure that the conclusion is also true, as long as the logic was done properly.

And that's what a lot of math really is. It's a long chain of "well if this is true, then this next thing definitely is as well". Mathematicians over thousands of years have worked to make math follow this structure, and any problem that can be solved with deductive logic can be looked at as a math problem. Math actually becomes very useful when the right starting statements are chosen - statements that *are* true and *do* connect with the real world properly.

Terms

Axioms

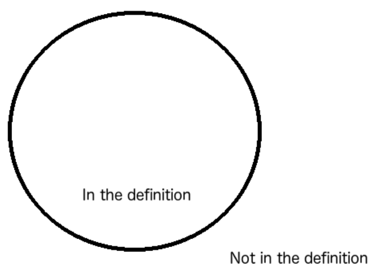
Axioms are the basic starting statements in math. Axioms are statements that are so basic, that there is no way for you to prove them - you can only believe that they're true, or choose not to. Things that you cannot find a proof of are not necessarily axioms. Axioms need to be extremely simple, and are not proved only because you can't have anything simpler to prove the axiom with.

When mathematicians disagree on what the axioms are, then building math from the ground up with these axioms will give different conclusions. Changing the meaning of the starting statements also changes the meaning of the conclusions. Like we said above, the most useful math comes from starting statements that connect well with reality. Mathematicians have made an effort to come up with meanings of axioms that are useful.

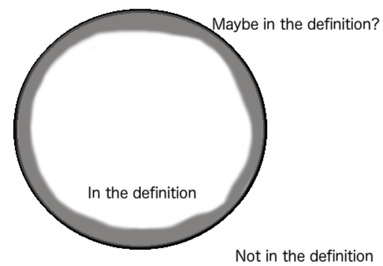
Definitions

A definition in math is just like what it is in language: it tells you the exact meaning of something. You do not have to prove a definition, because it doesn't actually say anything. A definition just tells you what you would mean if you said a statement. An important exercise in math is making sure things fit a definition.

An important part of creating definitions is making sure that the definition is "well defined". "Well defined" means that it's clear what is and is not a part of the definition. There should be no grey area, where you're not sure.



GOOD: Anything you choose should fit nicely inside or outside the definition.



NOT GOOD: If there are things that your not sure about, then the definition isn't well defined.

Theorems

A theorem is any other statement in math that needs a proof. Every time you state a new theorem, you need to use other things you already know, like axioms or theorems you've already proved, to prove the new theorem you're stating. There are other words used in math (like proposition or lemma) that mean the same thing, but are chosen depending on how important or new they are, or what they tell you. In this lesson, we'll just use the word theorem or statement.

Statements made in math that are not yet proved are often called conjectures or hypotheses. Some of these are actually very important to modern math, and are some of the biggest unsolved problems in math today.

Types of Statements

There are different types of statements you can make, and each of them needs a different approach for you to prove them.

Existential Statements

An existential statement is a statement that only tells you *that something exists*. It doesn't tell you anything more than that. The full statement follows a structure like:

there exists a *thing* such that this statement about it is true

For example, the statement "there exists a dog such that the dog has three legs". In regular English, we would probably just say that "there's a dog that has three legs". For this lesson we'll use the more strict structure above, just so it's clear what kind of statement we're making.

Universal Statements

A universal statement is a statement that tells you that *something is always true*. The full statement follows a structure like:

for all *things*, any *thing* you choose will have this statement true about it

For example, the statement "for all dogs, any dog you choose will have three legs". In regular English, we would probably just say that "all dogs have three legs". Again, for this lesson we'll use the more strict structure above, just so it's clear what kind of statement we're making.

Implications

An implication is a structure that a statement can have. Implications are also sometimes called "If... Then..." statements, since that's how they are usually read:

if this first statement is true, then this second statement is also true

In other words, the first statement *implies* the second one. For example, the statement "if a dog has three legs, then the dog has grey fur" is an implication.

Doing Proofs

Trying to actually prove theorems can be very difficult. Being such an important part of math, mathematicians have come up with techniques and approaches to take for proving the different types of statements.

Proving Existential Statements

To prove an existential statement, the idea is straight-forward: To prove the existence of something, you just have to find it.

1. Find a specific *thing*

2. Show that the statment is true for this *thing*

Step one of this process can sometimes be harder than you'd think. If there are only a few things to choose from to begin with (for example: marbles in a bag), then using trial and error is simple, and not a bad idea. If you have a lot of things to choose from (for example: any marble in the world), then you have to narrow down where you're looking first. As long as you find one thing though, your theorem is still proved. All you need to show is that it exists, and fits the statement.

Proving Universal Statements

To prove a universal statement, we cannot choose a specific thing to look at. Instead, we have to look at *things in general*.

1. Think about a general *thing*

2. Show that the statment is true for this general *thing*

For example, if my theorem is "for all dogs, any dog you choose will have three legs" then I have 2 choices. Since the things I want to look at are "all dogs", I can either (1) look at every single dog individually and show that all of them have three legs, or (2) look at dogs in general, and somehow show all at once that all dogs have three legs. In this example, we don't have enough information yet about dogs to prove this statement.

Let's say we also know that for any dog, the number of legs it has is related to the number of T-Rexes on Earth like this:

$$\text{(the number of legs a dog has)} = \text{(the number of T-Rexes on Earth)} + 3$$

*note: In math, we would usually use variables in an equation like this

This way, we're not talking about the number of legs of any specific dog, but of dogs in general. Let's say we also know that there are 0 T-Rexes on Earth. Then from the equation,

$$\text{(the number of legs a dog has)} = 0 + 3 = 3$$

which proves our universal theorem.

Proving Implications

To prove an implication or an "If... Then..." statement:

1. Assume that the "If..." is true
2. Show that the "Then..." also needs to be true when the "If..." is

To prove the statement "for all dogs, for any dog you choose, if the dog has three legs, then the dog has grey fur", you need to use the technique of proving universal statements. Look at dogs in general, and then show that if you assume the dog has three legs, then it also must have grey fur.

To prove the statement "there exists a dog such that if the dog has three legs, then the dog has grey fur", you need to use the technique of proving existential statements. Look a specific dog, and show for this specific dog that if you assume the dog has three legs, then it also must have grey fur. This example may not make complete sense if you think about what the words here mean, but that doesn't change how the deductive logic of the statements work. As long as the deductive logic is working properly, then the math being done is right.

Note:

Keep in mind, these are not all the proof techniques, and not the only types of statements you can make in math. They are, however, an important starting point before getting into more complicated techniques and mathematics.

Problems

REVIEW

1. What is an axiom? What is a Theorem? What is a definition?
   See the heading "Terms", on pages 2 and 3.

2. What are the two big branches of logic? What kind of logic does math usually use?
   The two big branches of logic are deductive logic, and inductive logic. Math usually uses deductive logic.

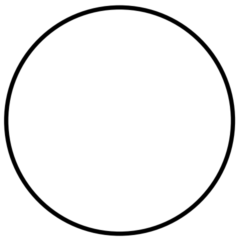
3. What does an existential statement tell you? What does a universal statement tell you? Give an example of each.
   An existential statement only tells you that there is a *thing* that exists. It also might give a property that this *thing* has. In other words *"There exists a thing, such that the thing makes this statement true"*. The property the *thing* has is in the statement that it makes true. An example is the existential statement "There exists a ball such that the ball is red". In this case, the *thing* is a ball, and the property they have is being red.

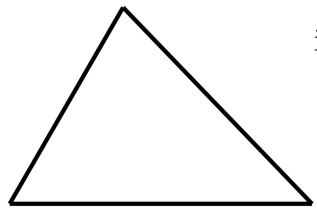
   A universal statement tells you that any *thing* in a group of *things* has a property. In other words, *"for all things, any thing you choose will make this statement true"*. The property is the statement that the *things* makes true. An example is the universal statement "For all balls, any ball you choose will be red". In this case, the *things* are all balls, and the property they have is being red.

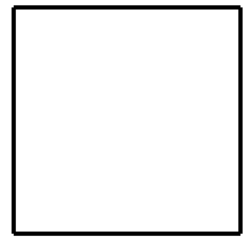
4. Give an example of an implication.
   If you do the Math Circles problems, then you will understand the lessons better.

APPLY

5. (a) Let's say the definition of a triangle is: a polygon with 3 straight sides. Using this definition, which of these are triangles?

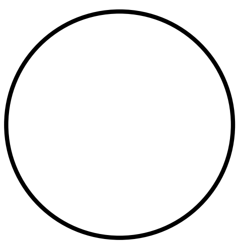
i. 

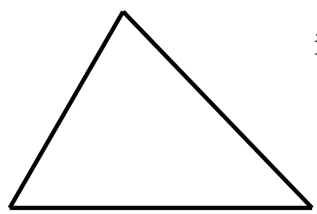
ii. 

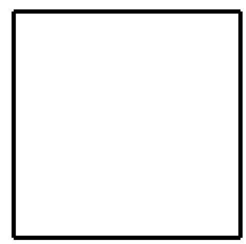
iii. 

ii. is the triangle, since it is a polygon with 3 straight sides, fitting the definition.

(b) Let's say the definition of a triangle is: a shape with no straight sides. Using this definition, which of these are triangles?

i. 

ii. 

iii. 

i. is the triangle, since it fits the definition of being a shape with no straight sides. Are there any more triangles you can draw that fit this definition?

6. Let's say the definition of an object being wet is that there is water on it. Based on this definition, is the water wet? For example, is the ocean wet? Do you think this is a good mathematical definition?

This definition of something being wet is not *well defined*. Is liquid water on itself? The answer is uncertain, and might depend on what you mean by "on". Words that are usually of scientific interest can be hard to define in a mathematical because of the difference in logic.

7. Goldilocks loves porridge. To her, porridge that is "just right" is defined as:

   - Between 12°C and 16°C
   - Made by a bear

   Papa Bear has a bowl of porridge that is 25°C, Mama Bear has a bowl of porridge that is 5°C, and Baby Bear has a bowl of porridge that is 15°C. They were all made by Mama Bear.

   (a) Use the definition to prove the theorem that there is exists a porridge such that the porridge is "just right".
   We know that at least three bowls of porridge exist from the question: Mama Bear's, Papa Bear's, and Baby Bear's. For these to be "just right", they have to fit the definition. We know that Mama bear made all three bowls of porridge (in other words "for all bowls of porridge stated in the question, any bowl of porridge you choose was made by a bear"). Baby Bear's porridge is also between 12° and 16°. Therefore Baby Bear's porridge exists and fits the definition, so the theorem is true.

   (b) Goldilocks is human, with a human grandmother. Prove the theorem that: for all porridge, if the porridge was made by her grandmother, then the porridge is not "just right".
   This theorem is a universal statement with an implication, talking about all porridges. So we do the following:

   - Consider a porridge, any porridge.
   - Assume that for this general porridge, the "if..." is true. In other words, assume that this porridge was made by Goldilocks' grandmother
   - Porridge that was made by her grandmother was not made by a bear, since her grandmother is human.
   - Porridge that is "just right" needs to be made by a bear, so any porridge made by Goldilocks' grandmother does not fit the definition of "just right".

   This proves the theorem by considering a general porridge, and therefore proving the statement for all bowls of porridge at once.

8. Consider this set of numbers: $\{2, 4, 6, 8, 10\}$. Let's say the definition of a number being even is that it is divisible by 2.

   (a) Prove that for all numbers in the set, any number you choose will be even.
   Since we only have a few numbers in the set, we can prove this universal theorem easily by proving that each number in the set is even individually. $2 \div 2 = 1$, $4 \div 2 = 2$, $6 \div 2 = 3$, $8 \div 2 = 4$, $10 \div 2 = 5$. Each of the numbers in the set is divisible by 2, so they are all even. This proves our theorem that for all numbers in the set, any number you choose will be even.

   (b) Prove that there exists a number in the set such that this number is not double another number in the set.
   This is an existence theorem, so we only need to show that one thing exists that makes the statement true. 6 exists and is in the set. $6 = 2 \times 3$ means that 6 is double 3. 3 is not another number in the set. This proves the theorem.

   (c) Prove that for each number in the set, for any number you choose there exists a whole number not necessarily in the set such that this whole number is half the number in the set that you chose.
   This universal theorem has an existential statement inside it.

   Since the set is small, we can prove the universal theorem by looking at each thing in the set individually. For each individual number in the set, we need to prove the existential statement that "there exists a whole number not necessarily in the set such that this whole number is half the number in the set that you chose".
   Doing $2 \div 2 = 1$, we get the whole number 1, which exists and is not necessarily in the set, and is half of 2, the chosen number from the set. This proves the *existential statement* for the number 2 from the set. We need to repeat this for every other number in the set to prove the whole universal theorem.

   We can also prove this universal theorem all at once by using the theorem we proved in 8 (a). Let's think about a general number, any number, from the set. We know from 8 (a) that all the numbers in the set are even numbers. By definition, a number being even means that it's divisible by 2. This means that by definition: *(the number from the set)* $\div 2 =$ *(a whole number)*. This means that for all numbers in the set, any number we choose will have a whole number that is half its size and is not necessarily in the set. This also proves the theorem, and is more efficient than the first proof since it uses statements that we've proved before.

(d) Prove that there exist a whole number not necessarily in the set such that for each number in the set, any number you choose will be a multiple of the whole number.

This is an existential theorem that has a universal statement inside it.

What we're looking for now is just one number that every number in the set is a multiple of. In other words, we're looking for a common divisor of the numbers in the set - a number that all the numbers in the set are divisible by. We can either find this number by trial and error, or we can use what we already know. Using 8 (a), we know that every number in the set is even. This means by definition that every number in the set is divisible by 2. Therefore 2 is a whole number that exists, such that for all numbers in the set, any number you choose is a multiple of it. This proves the theorem.

Notice that the number 2 is in the set. This is perfectly fine, because the theorem only say *"not necessarily in the set"*. If the number we find to prove the existential theorem is in the set, that doesn't contradict with the theorem at all.

CHALLENGE

9. A common proof technique in math is *proof by contradiction*. In a case where a *thing* has to be one of two things, a proof by contradiction proves a statement about the *thing* by showing that the opposite of the statement contradicts something else that you know is true. This works because we want math to be consistent. We don't want to say two statements are true if they say opposite things.

   As a simple example, let's say I want to prove 3 (my *thing*) is an odd number. I can start by assuming 3 is an even number. Using the definition of even we used before, 3 being even means 3 is divisible by 2. But, doing $3 \div 2 = 1.5$ shows 3 is not divisible by 2. This is a contradiction. Because I did everything else right, my first step of assuming 3 is even must have been wrong, so 3 must be an odd number.

   (a) Use proof by contradiction to prove 7 (b) above

   To do 7 (b) using proof by contradiction, we assume that the opposite of the theorem is true and show that's a contradiction. The opposite of the statement "for all porridge, if the porridge was made by her grandmother, then the porridge *is not* 'just right"' is "for all porridge made by her grandmother, the porridge is also 'just right"'

- Assume for all porridge made by her grandmother, the porridge is also "just right", looking for a contradiction.
- Any porridge made by Goldilocks' grandmother is made by a human, since her grandmother is human.
- This means porridge made by her grandmother does not fit the definition of "just right", and so is not "just right"
- This contradicts the original assumption that all porridge made by her grandmother is "just right". This means the original assumption must be wrong, so any porridge made by her grandmother *is not* "just right"

This proves the theorem.

(b) Take the following as axioms:

(1) For all babies, if a baby is not asleep then the baby is either crying or drinking milk.

(2) For all babies, for any baby you choose there exists a toy called "its toy" that makes the baby not cry.

(3) For all babies, any baby you choose can only have "its toy" when the baby is awake.

Use proof by contradiction to prove that for all babies, if a baby has "its toy" then the baby must be drinking milk.

We assume that the opposite of our theorem is true: for any baby that has"its toy", the baby must *not* be drinking milk. In other words, any baby with "its toy" has "its toy" AND is not drinking milk.

- This is a universal theorem so we will start by thinking about a baby, any baby.
- Axiom (3) tells us a baby can only have its toy and be awake at the same time. So any baby with its toy is also awake.
- Axiom (1) says that when not asleep (in other words when awake), a baby is either crying or drinking milk.
- We assumed that a baby always has "its toy" and *is not* drinking milk at the same time. So axiom (1) tells us the baby must be crying, since it's not drinking milk while it has its toy.
- This contradicts axiom 2, which says "its toy" makes the baby not cry.

The original assumption that "any baby with 'its toy' has 'its toy' AND is not drinking milk" contradicts things that we already know are true. This means the assumption must be wrong, and its opposite is true. Its opposite is our original theorem that for all babies, if a baby has “its toy” then the baby must be drinking milk. This proves our theorem.