UNIVERSITY OF WATERLOO

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Centre for Education in
Mathematics and Computing

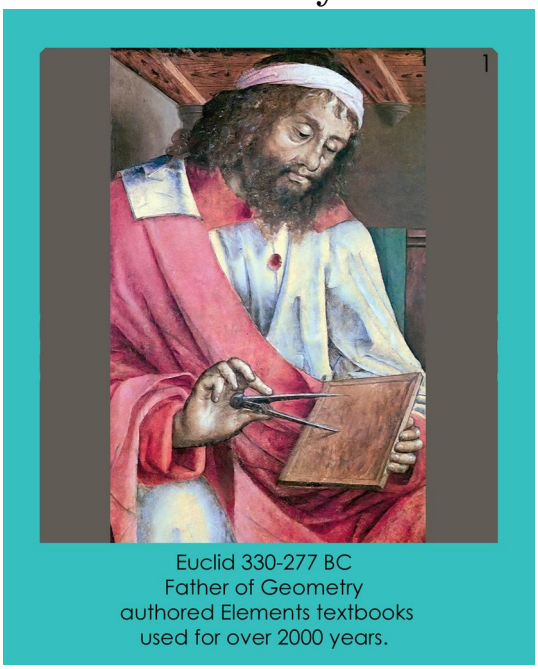
Grade 6 Math Circles

October 16 & 17 2018

Non-Euclidean Geometry and the Globe

(Euclidean) Geometry Review:

What is Geometry?



Geometry asks the question: what can we know about shapes?

2200 years ago, Euclid decided to answer that question, and wrote *Elements*. Almost every bit of geometry you will ever see done will have come from that book.

In *Elements*, Euclid made the rules for geometry using only two things:

1. Compass: The ability to make a circle by choosing a point as a center, and/or a second point to set the radius of the circle.

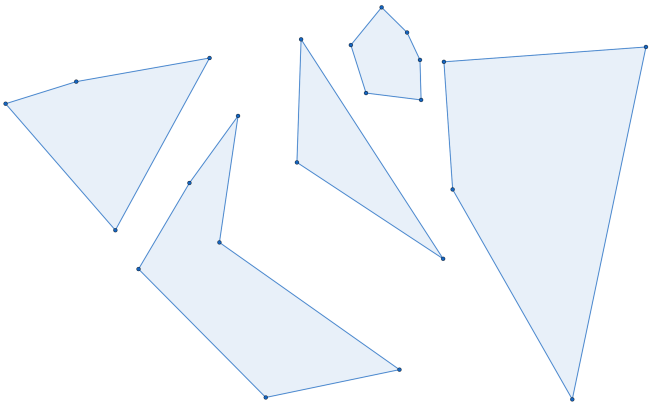
2. Straight edge: The ability to make a straight line using two points.

This is all geometry done on a flat piece of paper.

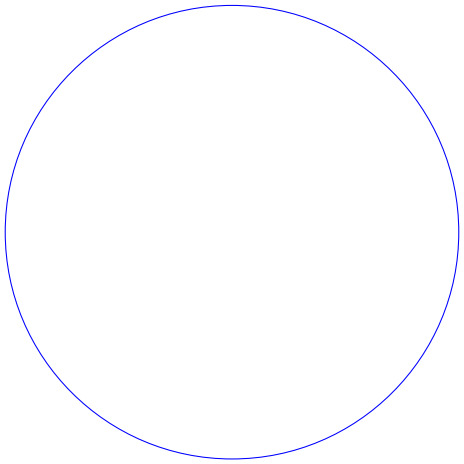
Euclidean Geometry Examples

With just a compass and straight edge, you can make:

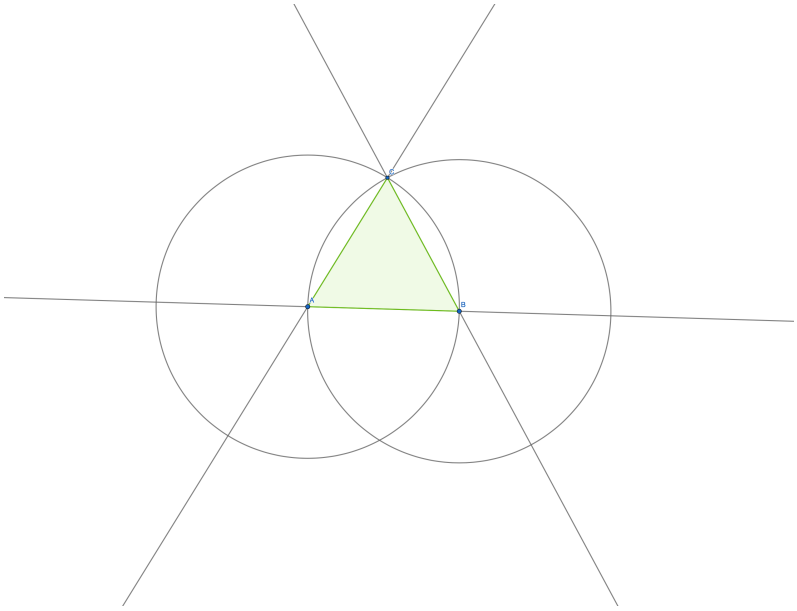
Polygons:



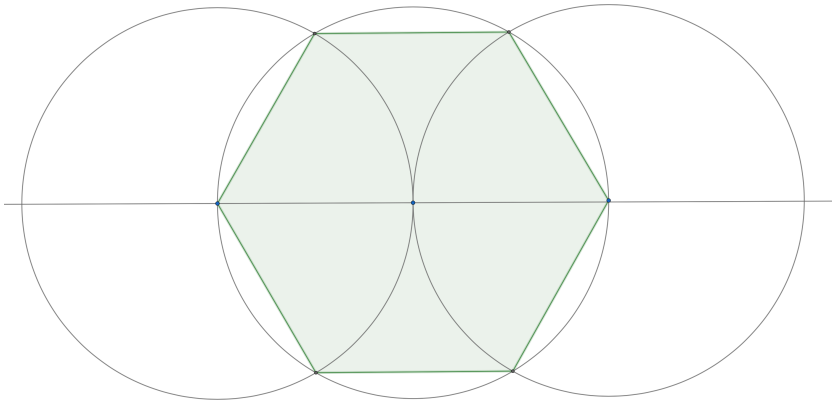
Circles:

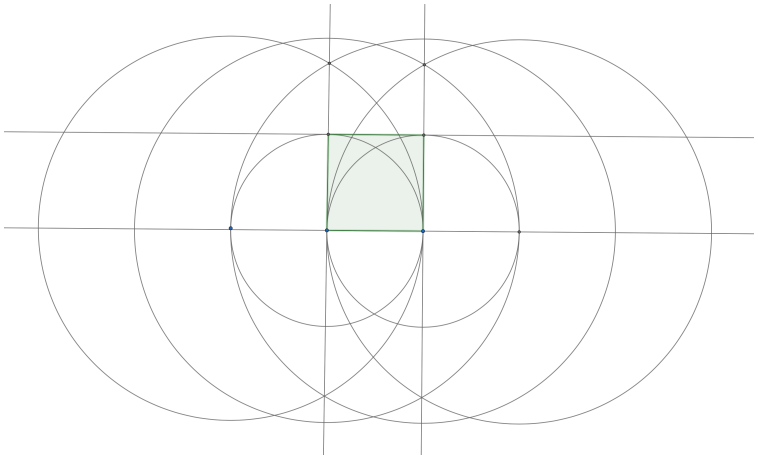


Equilateral Triangles:



Regular Polygons:

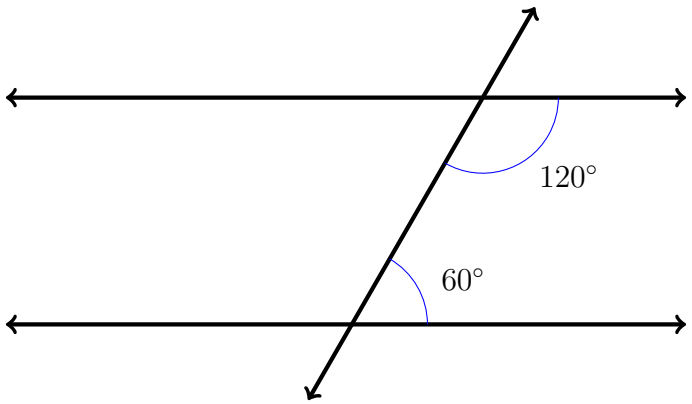




Parallel Lines

What are parallel lines? In *Elements*, Euclid defines parallel lines with his parallel postulate:

If two lines are intersected by a third line such that the sum of interior angles formed on one side is less than the sum of two right angles, then the original two lines will intersect on that side (if extended indefinitely).
If the interior angles formed sum to two right angles on both sides, then the original two lines will never intersect. These two lines in this case would be parallel.



$60° + 120° = 180°$ = two right angles

*Note: Try for yourself. The sum of interior angles for parallel lines is always 180°

Over time people have restated this in many ways. Another way of saying this was given by Playfair:

IF you have a point and a line,

THEN you can draw one and only one line going through the point that will never touch the original line. These two lines would be parallel

*Note: Both these definitions are talking about *straight lines* in particular.

This way of understanding parallel lines, called Playfair's axiom, means the exact same thing as Euclid's parallel postulate. We will use Playfair's axiom later in the lesson.

Parallel Lines Examples

Are these red lines parallel?

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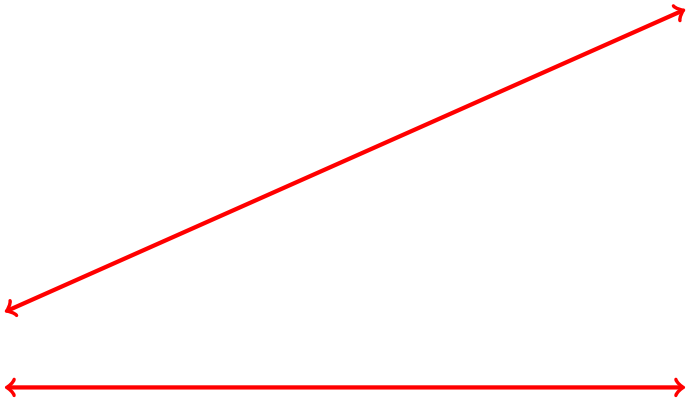
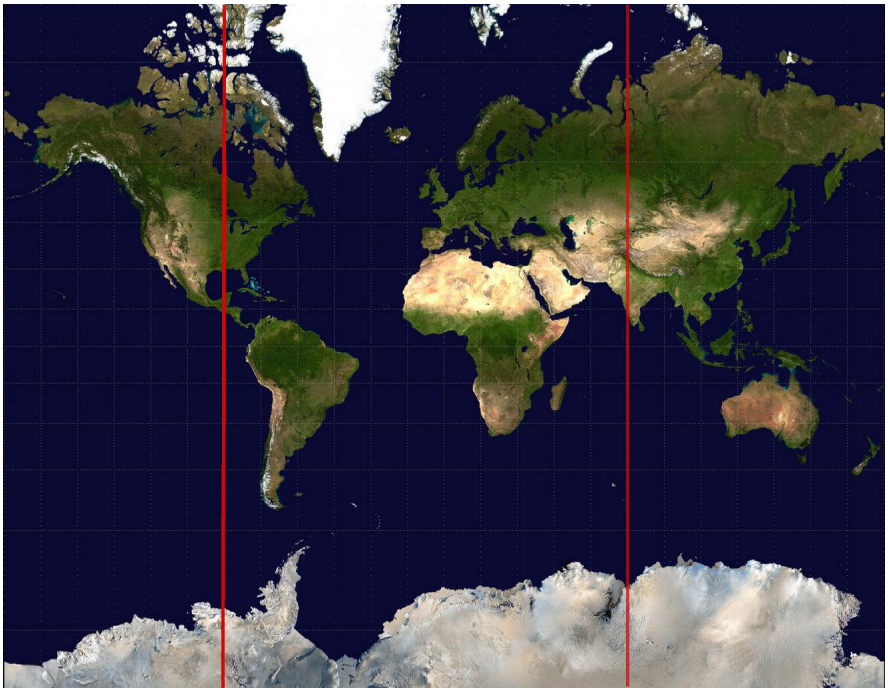
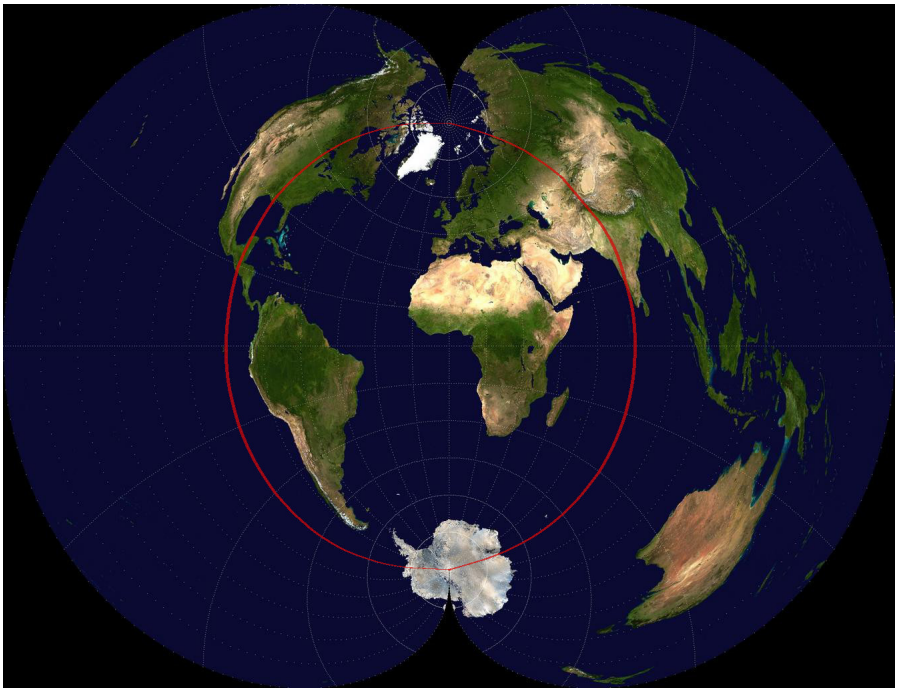


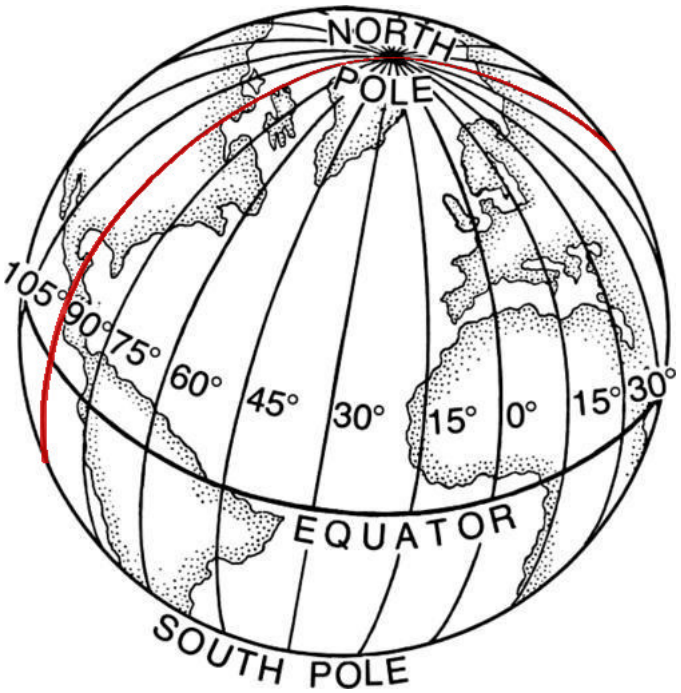
Are these red lines parallel on this map?



What if these same lines were on a different map?



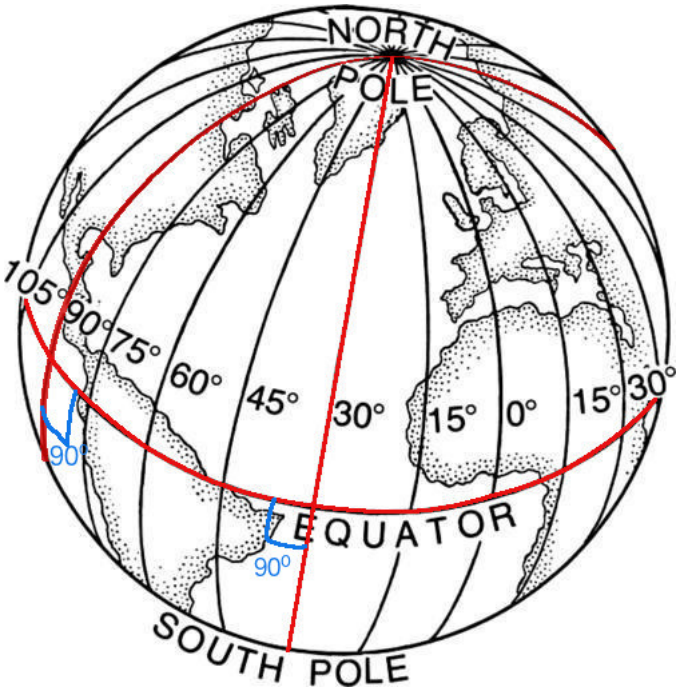
What if I put these same lines on a globe?



We're going to talk about globes as if they are *spheres*. Spheres are perfectly round 3D balls.

Look closely at the grid patterens on the two maps and the picture of the globe above. The red lines on these maps are exactly the same lines - it's just the maps that look different. Using Playfair's axiom, these lines are obviously not parallel since you can see where they touch at the North and South poles on the second map and on the globe.

As a matter of fact, we will see that Euclid's parallel postulate (and so also Playfair's axiom) is not even true on a globe. Notice that on the picture of the globe, the equator is perpendicular to both of the red lines. This means that the "sum of the interior angles" where the red lines are intersected by the equator is equal to two right angles (180°), but these lines are *still not parallel*. This tells us that the regular geometry we already know doesn't work if we're drawing on a globe. We need to do something different.



*Note: The above picture has an extra vertical line so you can see the interior angles better.

Non-Euclidean Geometry

You can think of non-Euclidean geometry as geometry that can't be done with a compass and straight edge in the way that Euclid would do it. In our case, we're talking about geometry on surfaces that are not flat (in particular spherical geometry on a globe).

Exercise: Triangles and Interior Angles

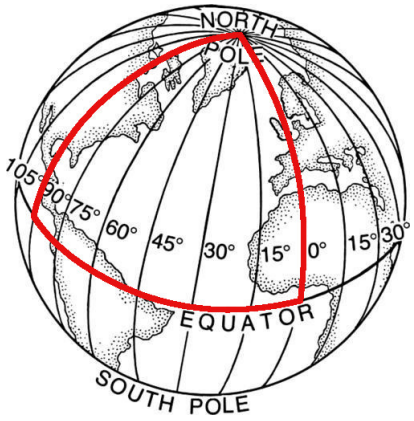
Let's review what a triangle is, and some of its properties:

- A triangle is three straight lines, coming together to make three vertices.
- The sum of interior angles of a triangle is 180°.
- The main types of triangles are: Equilateral, Isoceles, and Right Angle.

So, let's try drawing a triangle on the globe:

1. Start by drawing a line from the equator to the North pole of your globe.
2. From the North pole, 90° to draw a line that goes back to a different spot on the equator.
3. From here, draw a line along the equator that connects the lines you just drew.

You now in fact have an equilateral triangle.



What is the sum of the interior angles here? If you look at each of the vertices closely, you'll notice that you have three 90° angles. This means the sum of the interior angles of the triangle you just drew is 270°! This shows us again that geometry on a globe must be different from what it's like on flat paper.

Straight Lines and Curved Surfaces

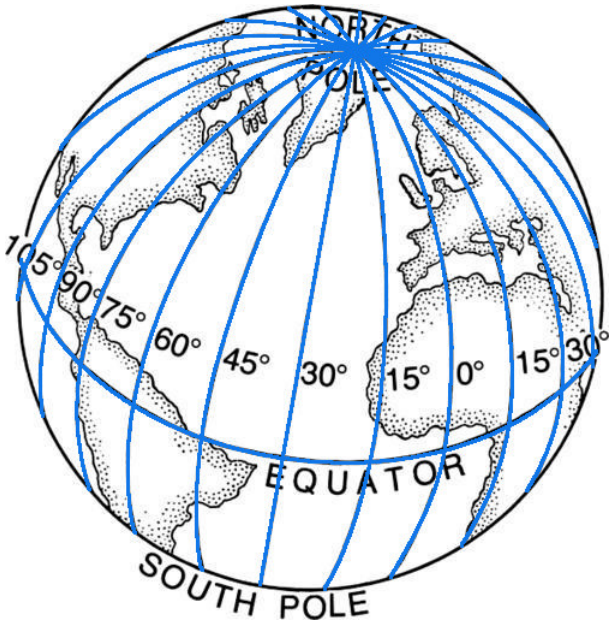
When we talked about straight lines and triangles in Euclidean geometry, we always said that we were using straight lines. Are the lines we've drawn on the globe straight? What does it mean for a line to be straight on the globe?

Think about this: what makes a straight line straight? What do straight lines *do* on a flat surface? Think back to *vectors* (covered last lesson): A straight line between two points on a flat surface is the shortest way to get from one point to the other. So on a curved surface, a straight line should do the same thing.

This is actually an important part of non-Euclidean geometry: what do you call a straight line in your space? From this we know what parallel lines and shapes are like in the space. The line that is the shortest distance between two points on a surface is called a ***geodesic***.

Straight Lines on a Sphere

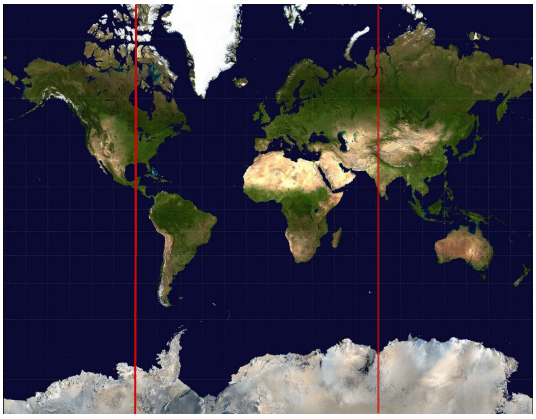
What do you think would be the shortest path to get from one point to another on a sphere? In other words: what is the ***geodesic*** on a sphere? It's actually a segment of a **great circle**: the biggest circle you can draw that goes all the way around the sphere.



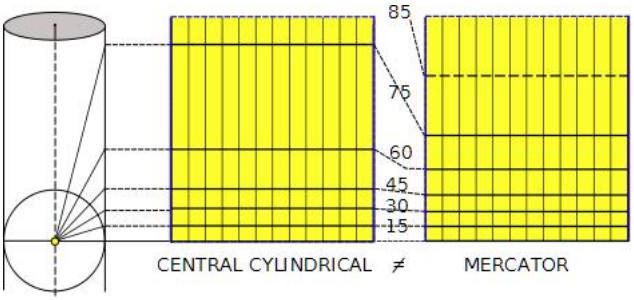
On the globe, the Equator is a great circle. Any line of longitude is also a great circle. The lines of latitude are NOT great circles, other than the equator. These are not the only great circles on a sphere, but they're the easiest ones to see on globe. This also shows us how Playfair's axiom is also false on a globe: ANY two "straight lines" on a globe will ALWAYS intersect.

Maps and Projections

Earlier in the lesson we talked about how the same lines on a globe can look different depending on the map you use to look at them. Why do you think these maps look so different? We know that the actual Earth is similar to a sphere, which is why we have globes. But a map is never a sphere. Maps are always drawn on flat pieces of paper. To be able to do this, mathematicians use ***projections***.



The map above, which is one of the most common maps, is made using the *Mercator projection*. The idea is to put the globe inside a cylinder, and make points on the cylinder according to a specific equation. Once you unroll this cylinder, you have a map!



What this means for us though is that we can't just draw lines on a map that look straight and call them parallel. Although a map is flat, any map is really just a *projection* of the globe. So the geometry that really matters in this case is the geometry that we can do on the globe.

Maps are also often made with a goal in mind. The Mercator projection was made to keep a 90° angle between lines of longitude and latitude. This was very helpful to navigators across the ocean, because it meant they could leave at a certain angle, and get to where they wanted to go without having to recalculate the angle.

In Summary

In the big picture, we learned:

- Euclidean geometry includes most geometry you'll see done in a flat space (and all you need is a compass and straight edge).
- Euclid's parallel postulate defines what parallel lines are on a flat surface. Playfair's axiom does the same thing, and is a useful way to understand parallel lines.
- Non-Euclidean geometry is just geometry that is not Euclidean. In particular: We looked at geometry on surfaces that are not flat

We looked more deeply at spherical geometry today:

- Math on a sphere (or any non-flat surface) can be very different from math on a flat piece of paper.
  - The properties we're used to shapes and lines having aren't always true on surfaces that aren't flat.
- Geodesic: Straight lines in spherical geometry are great circles
- We can use Projections on a globe to get a map, but doing that will always change something in how you're seeing the world.

Problems

REVIEW

1. What is the sum of the interior angles of a triangle in Euclidean geometry? What is the sum of the interior angles of an equilateral triangle in spherical geometry?

   The sum of the interior angles of any Euclidian triangle is 180°. The sum of the interior angles of a triangle in spherical geometry changes depanding on the triangle you draw, but it is always bigger than 180°. The equilateral triangle made between a pole and the equator (taking up $\frac{1}{8}$ of the area sphere) has a sum of interior angles of 270°.

2. In *Elements*, what two tools does Euclid use to do geometry?

   Euclid used a compass and a straight edge.

3. (a) Given a point and a straight line, how many straight lines can you draw through the point that are parallel to the first line on a flat piece of paper?

   Geometry on a flat piece of paper is Euclidean geometry. The question above describes the set up for Playfair's axiom:

   IF you have a point and a line,
   THEN you can draw one and only one line going through the point that will never touch the original line. These two lines would be parallel

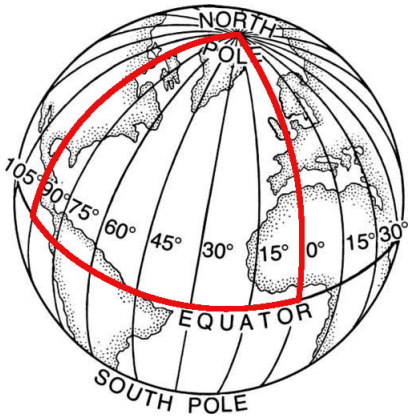
   which is true in Euclidean geometry. So you can only draw one line through the dot that's parallel to the first one.

   (b) Given a point and a geodesic line, how many geodesic lines can you draw through the point that are parallel to the first geodesic on a sphere?

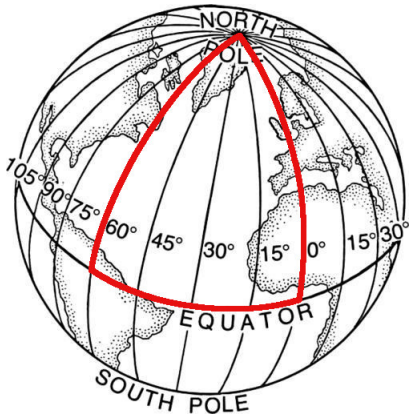
   Geometry on a sphere is non-Euclidean geometry, where Playfair's axiom is not true. Geodesics on a sphere are always great circles. Any two great circles on a sphere always intersect, so there are no geodesics that you can draw that are parallel to the first geodesic.

4. What are the sums of the interior angles of the following triangles?

(a)



(b)



(a) This is the equilateral triangle described in question 1. The sum of the interior angles here is 270°.

(b) We know that the angle between the Equator and any of hte lines of longitude is 90°, which gives us 2 of the angles. The 3rd angle we can easily read by looking at which lines of longitude the triangle has been drawn on. We can see that one side of the triangle has a line that goes along the 0° line up to the North pole from the Equator. The other side of the triangle goes along the line of longitude marked as 60°. The sum of the angles then is $90° + 90° + 60° = 180°$.

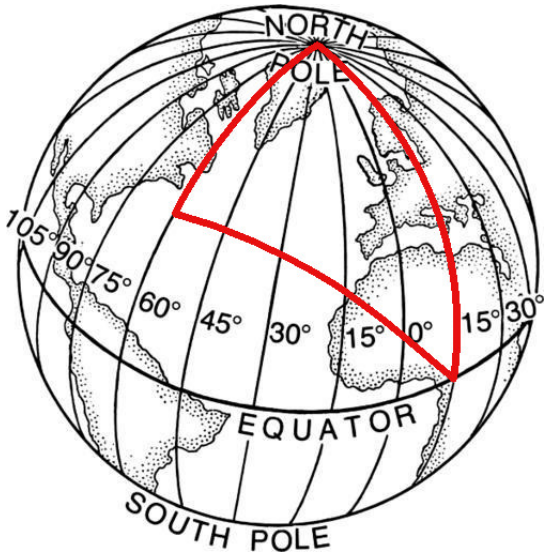
APPLY

The drawings below do not need to be done to complete accuracy, since measuring angles on a sphere can be hard.

5. If you can, go to https://www.euclidea.xyz/en/game/ to do some levels of geometry with only a ruler and straight edge.

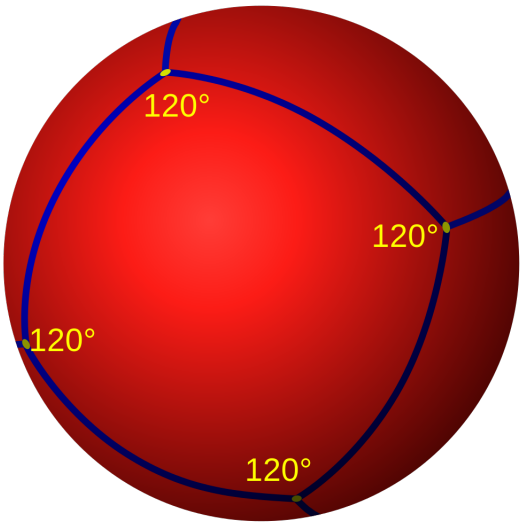
6. Draw a triangle on a globe that does NOT have a line on the Equator.

   The important part about drawing polygons on curved surfaces is that you're always drawing lines along geodesics. Even if a triangle on a globe doesn't have a line on the equator, as long as all the lines are along great circles the triangle is still drawn properly. As an example, a triangle may look like this:



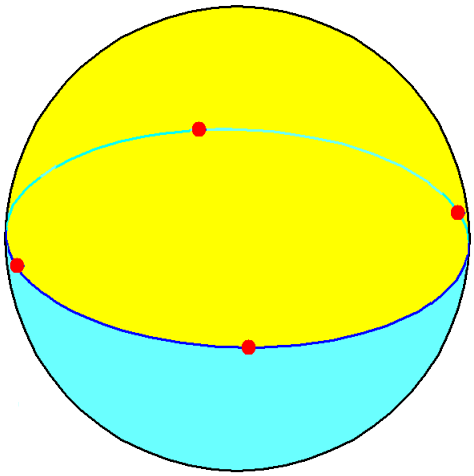
7. Draw a square on the globe. The sum of interior angles in a Euclidean square is 360°. Is the the sum of interior angles of a square in spherical geometry bigger or smaller?

   A square is a shape with equal side lengths and equal angles, drawn with straight lines. On a sphere, instead of straight lines we use segments of geodesics (which are great circles on a sphere). One example of a square on a sphere is:



   This square takes up $\frac{1}{6}$th of the sphere, and has a sum of interior angles of 480°

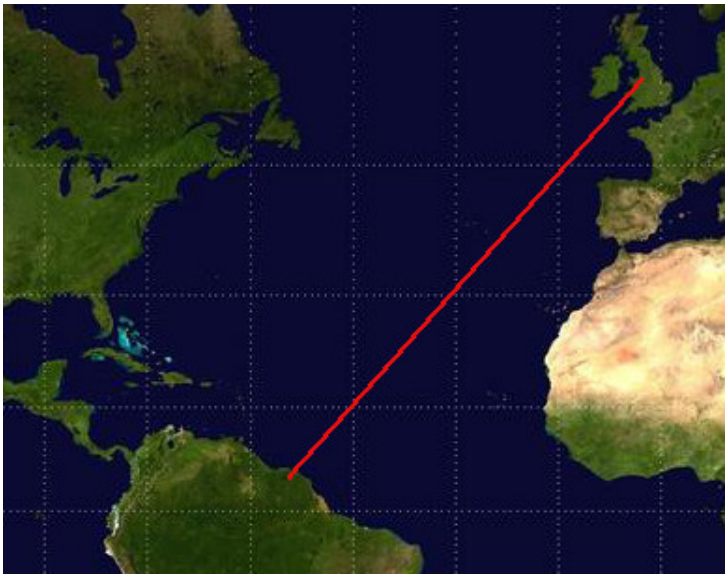
   Another example of a square on a sphere is:



By Tomruen [CC BY-SA 3.0 (https://creativecommons.org/licenses/by-sa/3.0)], from Wikimedia Commons

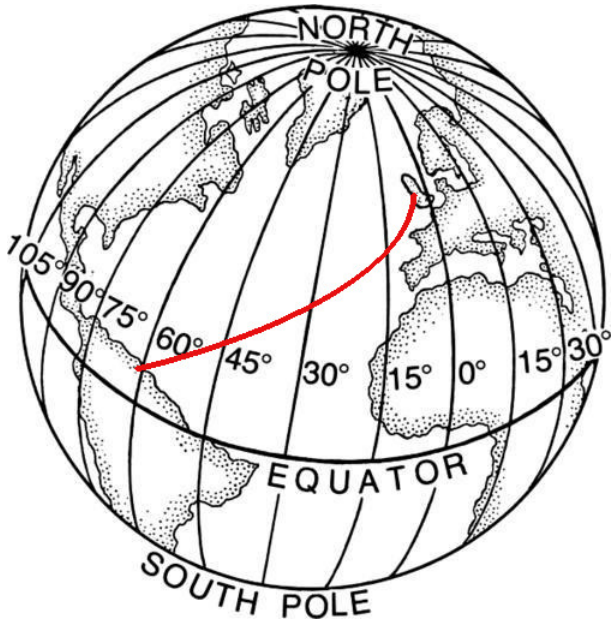
This sphere takes up $\frac{1}{2}$ of the sphere, and each angle internal angle is 180°, giving a sum of internal angles of 720°.

8. The Mercator projection was very useful for sailing across the ocean. The path that boats took across the ocean was a straight diagonal line on the Mercator projection. What do you think the path would look like on a globe? Would it be a straight line?



Looking at how the Mercator projection changes the way things look on the globe, we can imagine what this path actually looks like. Since the Mercator projection stretches things out more as you get closer to the poles, this line is actually curved on the globe,

and is more curved further from the equator. A rough guess of what it may look like is:



We can see that this path is clearly not a segment of a great circle. Since the geodesics of a sphere are great circles we can say that this path is not a "straight line" on the globe.

THINKING

9. Spherical geometry, which we talk about in this lesson, is a type of *elliptical geometry*. Elliptical geometry is a non-Euclidean geometry where

   IF you have a point and a straight line,

   THEN you can draw NO straight lines through the point that will never touch the original line.

   In other words, any geodesics you draw will always intersect, and there is no such thing as parallel lines. Can you think of any other types of non-Euclidean geometry? Think about how else you can change rules about parallel lines. Would the sum of internal angles of a triangle be bigger, equal to, or less than 180° in this new geometry?

Another type of non-Euclidean geometry is *hyperbolic geometry*. Hyperblic geometry is any geometry where

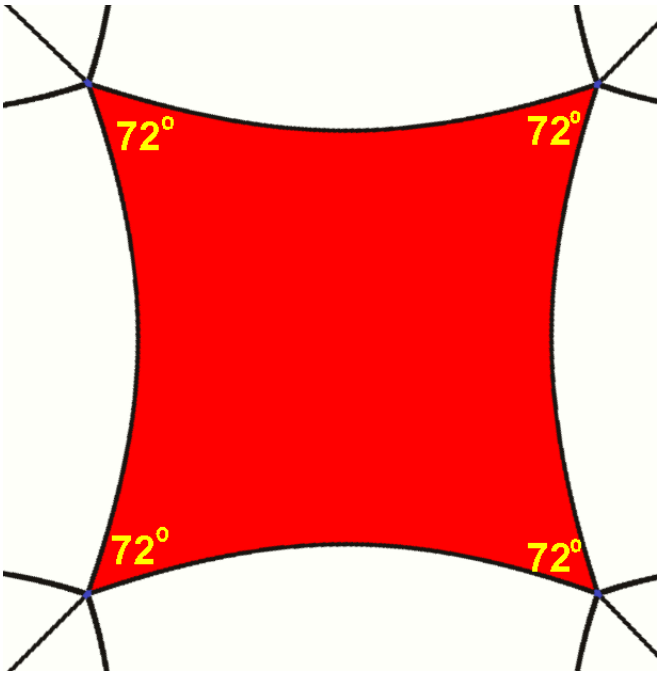
IF you have a point and a straight line,

THEN you can draw AT LEAST TWO straight lines through the point that will never touch the original line.

This is the only way left that we could modify Playfair's axiom, and it gives us a whole other type of geometry. In hyperbolic geometry, the sum of interior angles of a triangle will always be less than 180°, which is the opposite of spherical geometry.

Thinking about questions as simple as "what if these rules or properties were the opposite of what they are?" can give lots of interesting answers when doing math!

As an interesting extra, a square on a hyperbolic plane may look like this:



Remember that the lines here all need to be "straight", so these are all along geodesics in the space.