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# Grade 6 Math Circles

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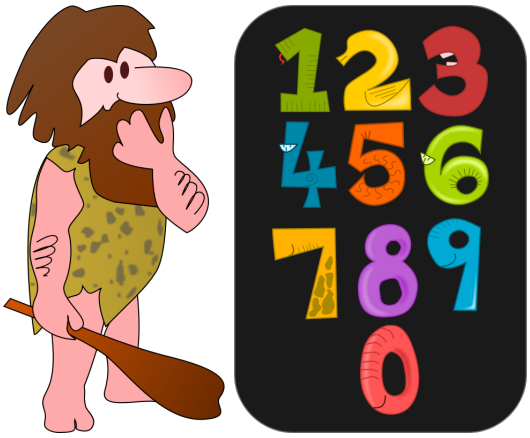
***Types of Numbers - Solutions***

## Introduction

Today, we take our number system for granted. We use numbers for everything: money, time, temperature, and so much more. Imagining a time without numbers seems impossible, yet numbers haven't been around forever.

Today, we will explore a little bit of the history of numbers, as well as the different types of number sets that have come to be over the course of discovering the numbers that we use today.



## History of Numbers

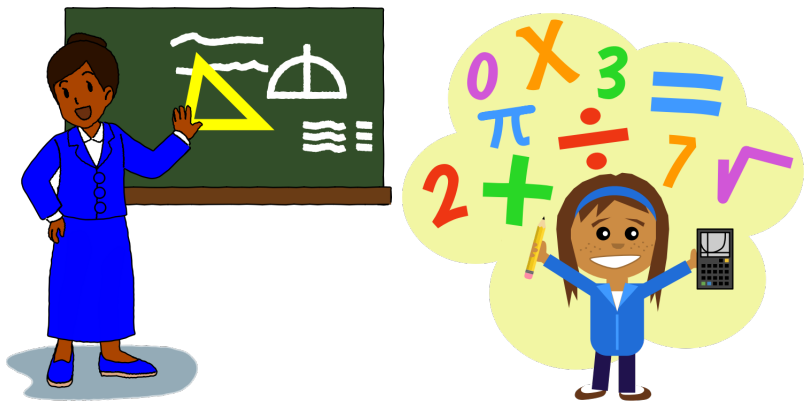
To understand the different sets of numbers, we should go back to a time when there were no numbers at all.

The earliest traces of numbers date back 150,000 years ago in Congo, when scratchings on a bone were found to be equally numbered on the front and back. In 4000 BC, Mesopotamians started using tokens for trading, which was the first sign of money. Then in Egypt, the first numerical system was developed where "one" was defined as the length between a man's elbow and fingertips plus the width of his palm. This is how counting began.

Most of our modern math originated in India, through the Arabic number system. They formally introduced whole numbers, and then used decimals for commerce, weight and measurements.

It was also in India where the concept of 0 started, first as a placeholder to distinguish numbers like 4 from 40. It wasn't until much later that 0 was thought of as a number: eventually, mathematicians wanted to answer the question "How can nothing be something?", which led to the invention of 0.

Some other notable mathematicians who contributed to the number system include the Greeks Pythagoras and Archimedes, who introduced geometry, and the Romans, who invented the numeral system that we are familiar with today.



## Natural Numbers

Although they used hieroglyphics to do it, the Egyptians were the first to count, so they are credited with the first numerical system.

The most basic set of numbers that we use is called the **natural numbers**, sometimes known as the whole numbers, or counting numbers.

This is the set {1,2,3,4,5,...}, or the set of positive whole numbers.

We don't include negatives, fractions, decimals, or 0.

Natural numbers are denoted by $\mathbb{N}$.

For example, 10000, 7, and 54 are all natural numbers, but -3, $\frac{1}{3}$ and 0.6 are not.

## Integers

Long before 0 was considered a number, negative numbers were used. The concept of negativity began in China in 200 BC when doing taxes, when they wanted to cancel out the amount sold with the amount spent to show a deficit. In India, negative numbers were used to show debts.

The set of numbers {...,-3,-2,-1,0,1,2,3,...} is known as the **integers**.
This set includes the natural numbers, the negative whole numbers, and 0.
Integers are denoted by $\mathbb{Z}$ (the Z means "Zahlen", which is German for numbers).
Since we have that natural numbers are in the set of integers, we say that natural numbers are a **subset** of integers, denoted in set notation as the following:
{Natural Numbers} $\subset$ {Integers}
For example, -72, 42 000 and 5 are integers, but -5.7 and $\frac{8}{5}$ are not.

# Rational Numbers

So far, we have looked at whole numbers - both positive and negative, as well as 0. However, we have not looked at the numbers in between the whole numbers. It was the Indians in 458 who first used the decimal system, when working in astronomy. Of course, you know that any decimal can easily be converted into a fraction.
Now that we add in fractions, we have a whole new set. We call the **rational numbers** the set of numbers that can be written as fractions. This means that we can write a rational number as a **ratio** of two integers, or a fraction with integers as the numerator and denominator (except 0 can never be the denominator of a fraction!).
Formally, we can write the set of rational numbers as $\{\frac{p}{q}$, where p,q $\in \mathbb{Z}$, q $\neq 0\}$.
($\in$ means "is an element of").
We denote rational numbers by $\mathbb{Q}$, which stands for "quotient" (since a fraction is a division).
Remember, all integers can be represented as a fraction with 1 as the denominator.
So, integers are a subset of rational numbers. We have:
{Natural Numbers} $\subset$ {Integers} $\subset$ {Rational Numbers}
Examples of rational numbers include $\frac{1}{3}$, $\frac{7}{6}$, $\frac{-2}{5}$ and $\frac{9}{1} = 9$.

# Irrational Numbers

Type $\sqrt{2}$ into your calculator. What do you get?
1.414213562...
Now try to represent this as a fraction. What do you find?
$\sqrt{2}$ can't be represented as a fraction.
The Greek mathematician Pythagoras believed that all numbers were rational. However, one day, one of Pythagoras' students, Hippasus, proved him wrong when he found that $\sqrt{2}$ could not be written as a fraction.
When a number is not rational, it cannot be written as a fraction. In terms of decimals, this means that it is never-ending! Notice that there is no pattern to the decimals of $\sqrt{2}$. If your

calculator was bigger, it would show a sequence of many more digits that would never end. We call the numbers that are not rational the **irrational numbers**, denoted by $\overline{\mathbb{Q}}$.

In set notation, we can say that rational numbers and irrational numbers are **disjoint**, which means that their sets have no numbers in common. To represent this, we write
{Rational Numbers} $\cap$ {Irrational Numbers} $= \emptyset$.
(the $\cap$ means the intersection of these sets, while $\emptyset$ means the empty set).
Examples of irrational numbers include $\pi$, $\sqrt{3}$, and the Golden Ratio.
However, numbers like $\frac{1}{3}$, which is equal to 0.333333333 (never-ending), are rational because the never-ending decimal is a repeating number. Only never-ending random decimals are irrational.
**All square roots of rational numbers are irrational, except for if the number is a perfect square.**

# Real Numbers

Once naturals, integers, rationals and irrationals had been created, mathematicians wanted a way to include all of these numbers in one set. So, they created the set of **real numbers**, denoted by $\mathbb{R}$. The real numbers are defined to be any number on the number line, or the set of rational or irrational numbers.
In set notation, we can say
{Natural Numbers} $\subset$ {Integers} $\subset$ {Rational Numbers} $\subset$ {Real Numbers}, and
{Irrational Numbers} $\subset$ {Real Numbers}.
Examples of real numbers include 11, 0, -42, 0.5, $\frac{9}{10}$ and $\sqrt{8}$.

## Practice

Determine which sets the following numbers belong to:

a) 4

Naturals, Integers, Rationals, Reals

b) 4.5

Rationals, Reals

c) -4

Integers, Rationals. Reals

d) $\frac{-1}{4}$

Rationals, Reals

e) $\sqrt{11}$

Irrationals, Reals

f) $\sqrt{4}$

Naturals, Integers, Rationals, Reals

## Imaginary Numbers

We have looked at all of the numbers on the number line. But what about numbers that are not on the number line?

What is $1^2$?

What is $(-1)^2$?

Is there a number $i$ such that $i^2 = -1$?

Since there are no real numbers that satisfy this, we define an **imaginary number** to be a number $i$ such that $i^2 = -1$.

We denote imaginary numbers by $\mathbb{I}$.

Examples of imaginary numbers include $i$, $3i$, $-9i$, and $\frac{3i}{4}$.

If we let $x^2 = -9$, then $x = \sqrt{-9} = \pm i\sqrt{9} = \pm 3i$.

## Practice

1. If $x^2 = -25$, what is $x$?

$x = \pm 5i$

2. If $x = \pm 2i$, find $x^2$.

$x^2 = -4$

# Complex Numbers

Now "imagine" combining a real number with an imaginary number. What would you get? For example, what is $3 + 4i$?

The answer is: $3 + 4i$.

We cannot combine real and imaginary numbers, so we leave them together and call them **complex numbers**.

A complex number has the form a + b$i$, where a is any real number and b$i$ is the imaginary part (b is any real number multiplied to $i$).

We denote complex numbers by $\mathbb{C}$.

We can also say that {imaginary numbers} $\subset$ {complex numbers} since an imaginary number is just a complex number with a = 0.

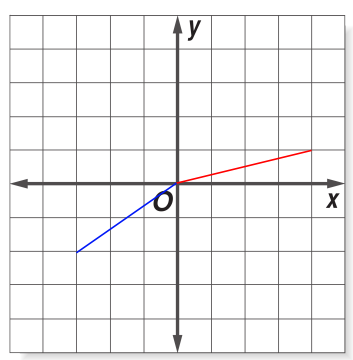
That means that {real numbers} $\subset$ {complex numbers}, if b = 0.

Examples of complex numbers include 7, $6i$, and $7+6i$.

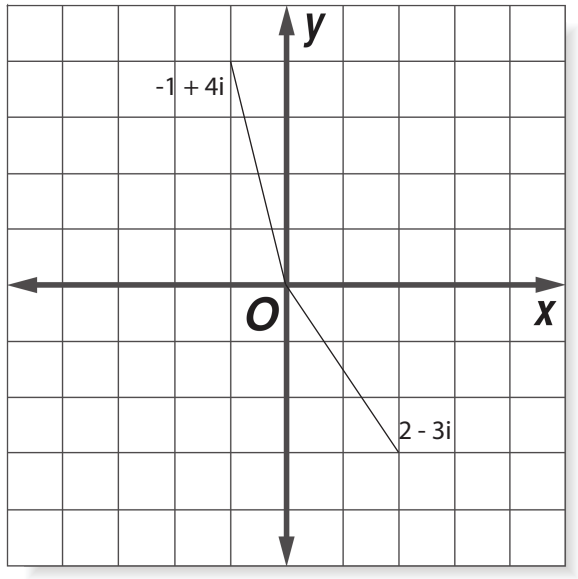
# Graphing Complex Numbers

Graphing complex numbers is far from complex. Let the x-axis be the real number line and the y-axis be the imaginary number line. Then, for a complex number a + b$i$, simply plot (a,b).

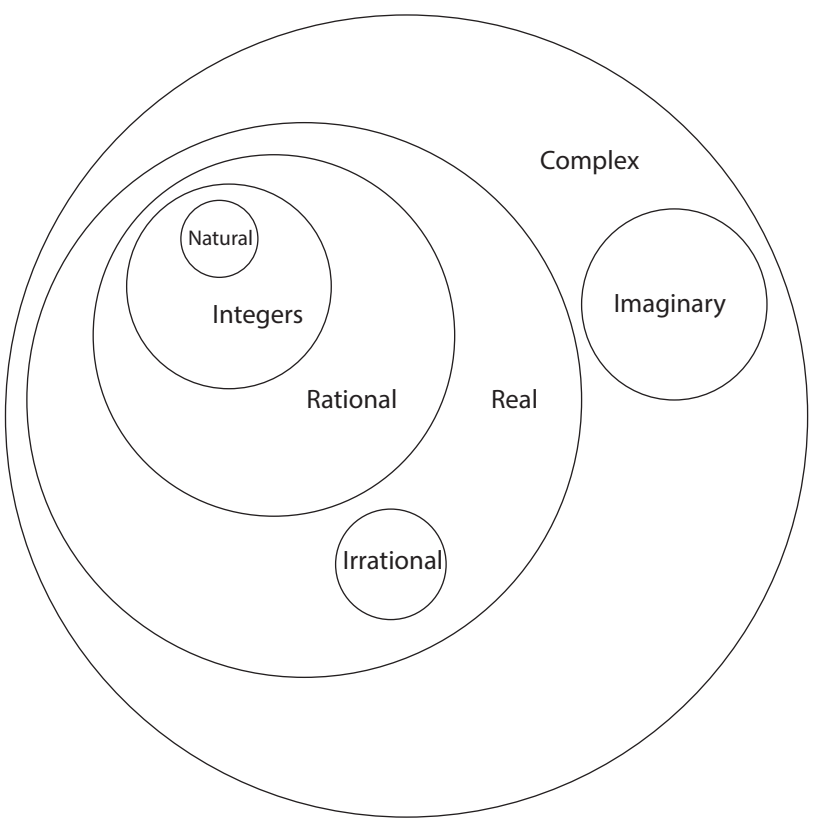
For example, the red line represents the complex number $4 + i$ since we plotted the point (4,1), and the blue line represents the complex number -3 - $2i$ since we plotted the point (-3,-2).



Draw the complex numbers -1 + 4$i$ and 2 - 3$i$.




# A Visual Representation of Types of Numbers



# Set Theory

We already learned that a subset is a set contained in another set.

We learned the term disjoint earlier using **intersection**, denoted by $\cap$. The intersection of sets A and B is the set of all elements that are in both of these sets (the elements in set A **and** set B).

For example, the intersection of the natural numbers and the integers is represented by the following:

$\mathbb{N} \cap \mathbb{Z} = \{1, 2, 3, ...\} = \mathbb{N}$

The **union** of two sets, denoted by $\cup$, represents all elements in at least one of the sets (the elements in set A **or** set B).

For example, the union of the natural numbers and the integers is represented by the following:

$\mathbb{N} \cup \mathbb{Z} = \{...-3, -2, -1, 0, 1, 2, 3, ...\} = \mathbb{Z}$

The **complement** of a set A is the set of numbers not in A. It is denoted by $\overline{A}$.

For example, the complement of the set $\{2, 3, 4, ...\}$ when looking only in $\mathbb{N}$ is $\{1\}$.

Also, the complement of the rational numbers is the irrational numbers.

$\bar{\mathbb{Q}} = \overline{\mathbb{Q}}$

Oh! That must be why we denote the irrational numbers by $\overline{\mathbb{Q}}$!

We have that **the intersection of any set and its complement is disjoint**: A $\cap \overline{A} = \emptyset$

# Practice

Evaluate the following:

1. $\mathbb{N} \cap \mathbb{I}$

$\emptyset$

2. $\mathbb{I} \cup \mathbb{C}$

$\mathbb{C}$

# Addition and Subtraction of Complex Numbers

Consider 2 complex numbers $z = a + bi$ and $w = c + di$. Here is how we add $z + w$:

$z + w = (a + bi) + (c + di)$

$= a + bi + c + di$

$= (a + c) + (b + d)i$

Just add the real parts together and add the imaginary parts together!

The same goes for subtraction:

$z - w = (a + bi) - (c + di)$

$= a + bi - c - di$

$= (a - c) + (b - d)i$

Let's try an example:

$(4 + 2i) + (1 + i)$

$= 4 + 2i + 1 + i$

$= (4 + 1) + (2 + 1)i$

$= 5 + 3i$

## Practice

1. $(10 + 7i)$ - $(6 + 3i)$

$4 + 4i$

2. $(1 + i) + (2i)$

$1 + 3i$

## Problems

1. What sets do the following numbers belong to?

a) $17i$

b) $13 + 3i$

c) $\pi$

d) 36

e) $\frac{-75}{17}$

f) 0

g) -72

2. Answer the following true/false questions. If the statement is false, give a counterexample.

a) The product of 2 irrational numbers is always irrational.

b) The product of 2 integers always an integer.

c) The product of 2 complex numbers is always complex.

d) The product of 2 natural numbers is always an integer.

3. Determine whether the following are rational or irrational.

a) $\frac{5}{6}$

b) -5.8

c) $\sqrt{99}$

4. Evaluate the following.

a) $(3 + 4i) + (7 + 8i)$

b) $(3 + 4i)$ - $(7 + 8i)$

c) $(-11 - 6i)$ - $(9 - 12i)$

d) $(0 + 0i) + (2 + 4i)$

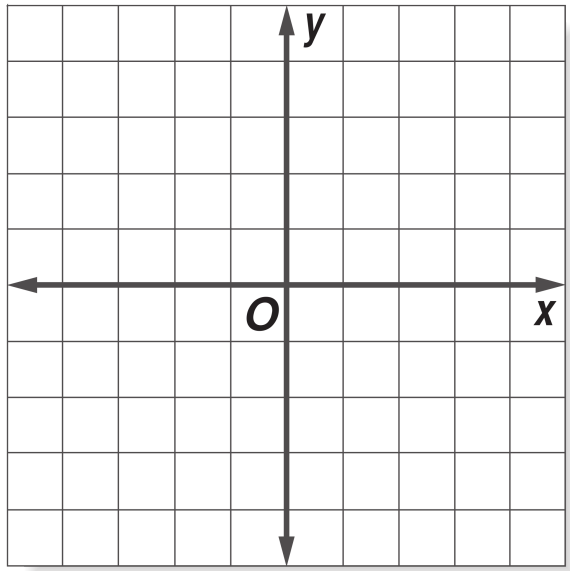
5. Plot the following complex numbers on the complex plane:

a) $(3 - 4i)$

b) $2i$

c) 5

d) $-4 + 5i$




6. Solve for $x$.

a) $x^2 =$ -64

b) $x^2 = \dfrac{-16}{9}$

c) $x^2 =$ -48

7. If $x = 36$ and $y = 4$, find if the following are rational:

a) $\sqrt{x+y}$

b) $\sqrt{x-y}$

c) $\sqrt{xy}$

d) $\sqrt{\dfrac{x}{y}}$

8. Evaluate the following:

a) $\mathbb{R} \cap \mathbb{I}$

b) $\mathbb{Q} \cup \bar{\mathbb{Q}}$

9. Evaluate the expression $(1 + 2i) \times (3 + 4i)$

10. The **modulus** of a complex number $z = a + bi$ is the magnitude of that number, denoted |z|. Using the plots you made from question 5, find the length of each line segment, then find a formula for the modulus of a complex number.

# Solutions to Problems

1. a) Imaginary, Complex

b) Complex

c) Irrationals, Reals, Complex

d) Naturals, Integers, Rationals, Reals, Complex

e) Rationals, Reals, Complex

f) Integers, Rationals, Reals, Complex

g) Integers, Rationals, Reals, Complex

2. a) False: $\sqrt{2} \times \sqrt{2} = 2$, which is rational.

b) True

c) True

d) True

3. a) Rational

b) Rational

c) Irrational

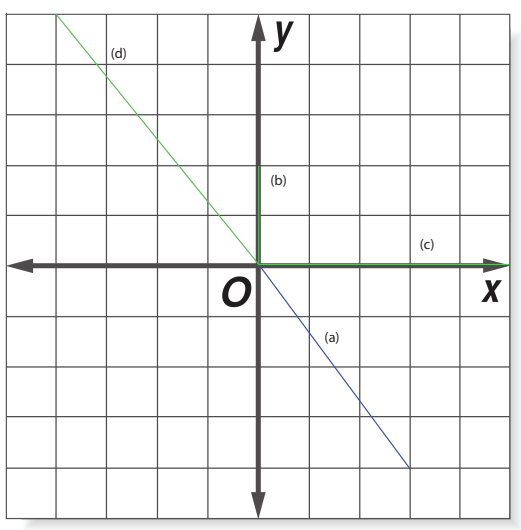
4. a) $10 + 12i$

b) $-4 + 12i$

c) $-20 + 6i$

d) $2 + 4i$

5.



6. a) $x = \pm 8i$
b) $x = \pm\frac{4}{3}$
c) $x = \pm 4\sqrt{3}$

7. a) $\sqrt{40}$ is irrational
b) $\sqrt{32}$ is irrational
c) $\sqrt{144} = 12$ is rational
d) $\sqrt{9} = 3$ is rational

8. a) $\emptyset$
b) $\mathbb{R}$

9.$(1 + 2i) \times (3 + 4i)$
$= 3 + 4i + 6i + 8i^2$
$= 3 + 10i$ - 8
= -5 + $10i$

10. Let $w = 3 - 4i$, $x = 2i$, $y = 5$, and $z = -4 + 5i$. We can easily see that $|x| = 2$ and $|y| = 5$. For the other two, imagine they are the hypothesis of a right triangle and use Pythagorean Theorem. Then we get $|w| = \sqrt{3^2 + (-4)^2} = 5$, and $|z| = \sqrt{(-4)^2 + 5^2} = \sqrt{41}$. So, a general formula for the modulus of a complex number is $|z| = \sqrt{a^2 + b^2}$.