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Grade 7/8 Math Circles

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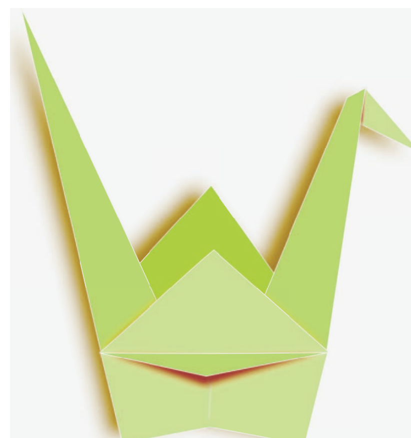
Spatial Visualization and Origami

Everybody loves origami! Origami is a traditional Japanese craft. *Origami* literally means "Folding Paper" with *Oru* in Japanese meaning "to fold", and *Kami* meaning "paper". It started in 17th century AD in Japan and was popularized in the west in mid 1900's. It has since then evolved into a modern art form. Today, designers around the world work with this exquisite art form to make all kinds of wonderous things! The goal of this art is to transform a flat sheet of material into a finished sculpture through folding and sculpting techniques. Cutting and glueing are not part of strict origami. A modified form of origami that includes cutting is called *kirigami*.



In today's lesson, we will explore the mathematical wonders of this sophisticated craft as well as focus on how this art form has helped in our capability to visualize geometric transformations in space.

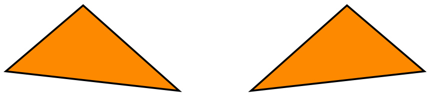
Definitions:

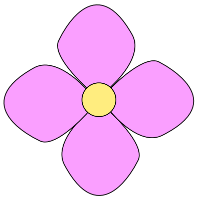
A **line of symmetry** is also known as the *mirror line* where an object looks the same on both sides of the "mirror".

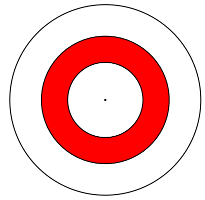
A **reflection** is a transformation where an object is symmetrically mapped to the other side of the *line of symmetry*.

Creases on a sheet of folded paper are the lines that you folded along after you open up your folded origami.

Examples Identify the number of lines of symmetry in the following figures.

(a) 

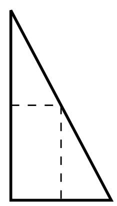
(b) 

(c) 

Identify Geometry Properties using Origami

Let's see some geometry first. Some properties of a geometric shape and formulas we use in calculating area of a shape can be easily explained.

1. Dividing the hypotenuse in half on a right angled triangle.



   (a) Fold along the hypotenuse such that the upper tip of your triangle touches the bottom of the hypotenuse, open it up, this point is half way of the hypotenuse.

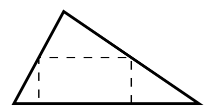
   (b) Fold along the bottom edge of your triangle such that the crease bisects this point

   (c) Fold along the side edge of your triangle such that the crease also bisects this point

   How does the resulting area of the smaller triangles compare to the big triangle?

   What relationship do you notice amongst the small triangles?

2. Rectangle half the height of the triangle.



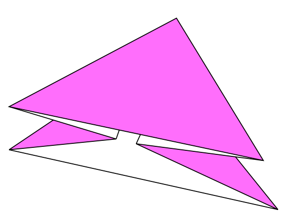
   (a) Suppose you have any triangle. Choose the longest side to be your base (This is to eliminate not seeing the rectangle on an obtuse triangle, for an acute triangle, you may choose any side to be your base).
   (b) Fold the top corner down so that it touches the base and the crease created is parallel to the base.
   (c) Fold in the right corner such that it makes a crease that is perpendicular to the base and intersects the previous crease at the right edge.
   (d) Repeat step (c) for the left corner.
   (e) To check that you are correct, you should get a rectangle formed by the 3 creases and the base of your triangle.

   For the rectangle enclosed by the creases on paper, what is the area of this rectangle compared to the big triangle? Can you explain why?

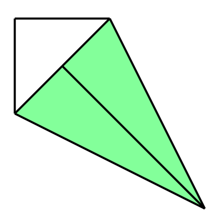
Colourability of Origami

A **flat fold** is a fold such that the resulting object can lie flatly on a surface (i.e. the resulting object is 2D).
Let's do the following investigation on flat folds. We will fold a water-balloon base as well as a kite base to illustrate the colourability of flat folds.

Water-balloon base



Kite base



Example

(a) Fold a water-balloon based figure (simple frog) as instructed and open up your fold.

Now the task is to grab your colour pencils and try to colour this using different colours such that no two adjacent pieces are the same colour:

Can you colour this using 4 different colours?
Can you colour this using 3 different colours?
Can you colour this using 2 different colours?

(b) Fold a kite based figure (swan) as instructed and open up your fold.

Can you colour this using 2 different colours?

Theorem

Any opened up origami paper that has the crease of a flat-fold can be coloured using just ____ colours.

Creases and Symmetry in Origami

A lot of origami folds are symmetrical on both sides without having you fold them twice. This is especially true when creating origami animals. For example, the famous paper crane is symmetrical.

Why is this so? Because what is done on one side is reflected along the lines of symmetry as you fold your paper.

Definitions:

An **inner most corner** is the corner(s) where you see the least number of layers of sheet paper after a series of folds.

An **outer most corner** is the corner(s) where you see the most number of layers of sheet paper after a series of folds.

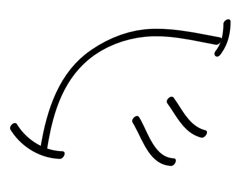
Examples

(a) How does the initial face become the transformed face (describe the transformations and the lines of symmetry)?

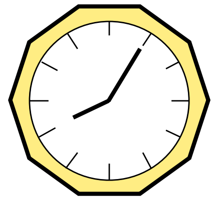
Initial face

Transformed face

(b) How does the initial face become this (describe the transformations and the lines of symmetry)?



(c) You are sitting facing a mirror and see this in the mirror. What time is it actually?



Definitions:

A **kirigami** is origami with cutting! In the strict definition of origami, cutting is not involved. But in kirigami we may fold and cut, for example, paper snowflakes!!! Yay snowflakes!

The **smallest component** is the smallest portion of a symmetrical figure such that it cannot be generated by reflecting a smaller component against a line of symmetry. (i.e. It is asymmetric) The entire figure is generated by repeatedly reflecting this component against multiple lines of symmetry.

Being able to visualize the crease and cuts after a series of folding and cutting helps to develop one's spatial sense. We will illustrate some simple techniques using lines of symmetry of origami and kirigami.

Tips/Strategy:

1. Identify the smallest component of your entire structure that is independent on its own. (i.e. the smallest component that cannot be constructed using reflection on some line from a smaller component)

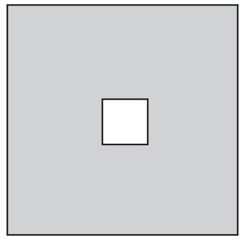
2. Remember, every time the paper is folded in half, twice the amount of cutting is saved. (Likewise, if you fold your paper in thirds, you save three times the amount of cutting, etc.)

3. In general, it is *easier* to FOLD than to CUT identical pieces.

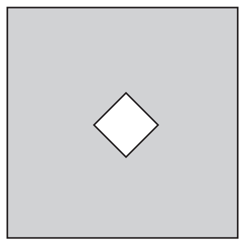
4. Whenever you cut after a fold, this cut becomes symmetrical on the other side of your fold line.

Examples

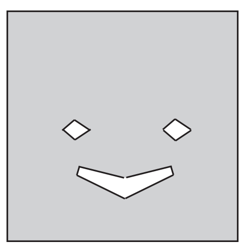
(a) I want to cut this paper exactly once to get the following image. How do I fold before I cut?



(b) I want to cut this paper exactly once to get the following image. How do I fold before I cut?



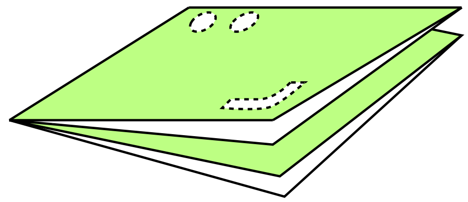
(c) I want to cut this such that the eyes and mouth are both symmetrical along the middle. My scissors cannot dig into the paper. How do I fold before I cut?



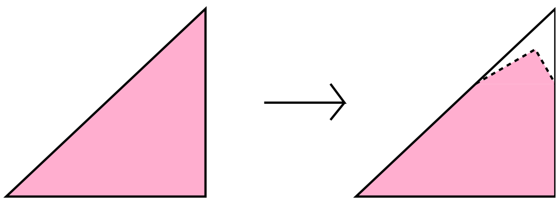
(d) I fold my square shaped paper in half, then in half again, creating a square a quarter of the original size. Then I fold the corner where I can see the sheets (outermost corner) towards the innermost corner, creating a triangle $\frac{1}{8}$ the original size. I open it, what do the creases look like?

(e) I fold my paper in half diagonally, then in half again, creating a triangle a quarter of the original size. Then I fold the innermost corner towards the bottom of the triangle (outermost edge) such that it just touches the edge, creating a trapezoid. Then I open it, what is the area of the square enclosed by the crease marks with respect to the original square piece of paper?

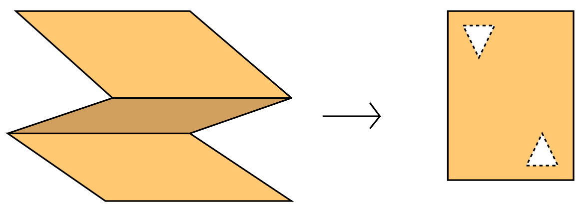
(f) I fold my paper in half, then in half again, creating a square a quarter of the original size. Then I cut it like this. I open it, what does the entire paper look like?



(g) I fold my paper in half, then in half again, creating a square a quarter of the original size. Then I fold this square in half again with the crease dividing the innermost corner, creating a triangle an eighth of the original size. Then I cut the inner most corner along the dotted lines. I open it, what does the entire paper look like?



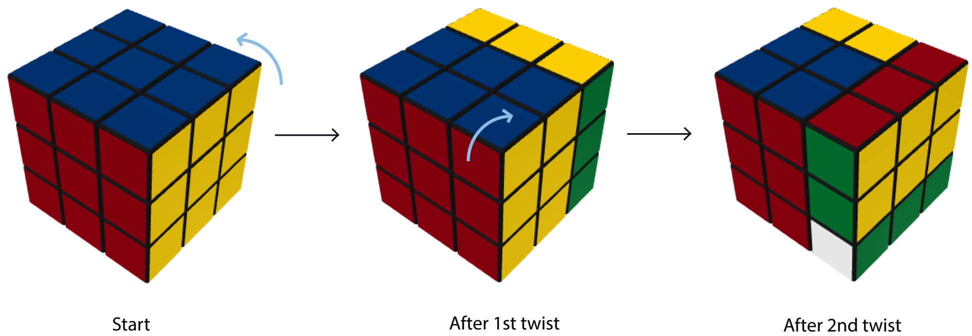
(h) I fold my paper into thirds, once over and once under. Then I cut it like this.



I open it, what does the entire paper look like?

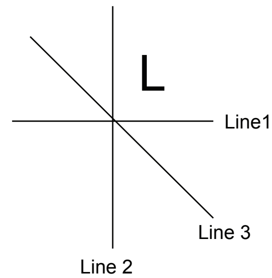
(i) This is Kingsten's Rubix cube. He twists the cube 3 times clockwise on different faces. What does Kingsten's Rubik's cube look like after the 3rd twist? The cube after the first and second twist is given below.

(**Note**: You do not need to know any other colours of the cube to draw the cube after the 3rd twist)

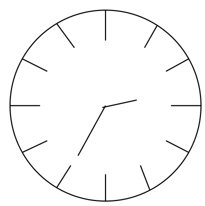


Problem Set

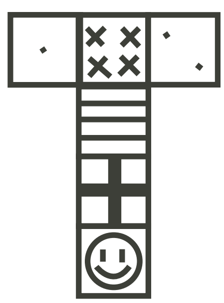
1. The "L" is reflected along line 1, then on line 2, then on line 3. What does the resulting "L" look like?



2. This is the time in the mirror, what time is it actually?

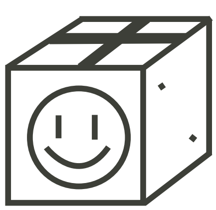


3. Grace bought an interesting piece of paper to assemble a cube. The unassembled paper is shown below. After the assembling the cube, is the configuration on the right possible?

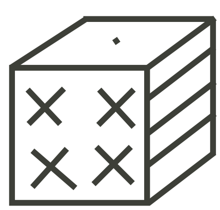


Are the following configurations of the cube possible?

(a)



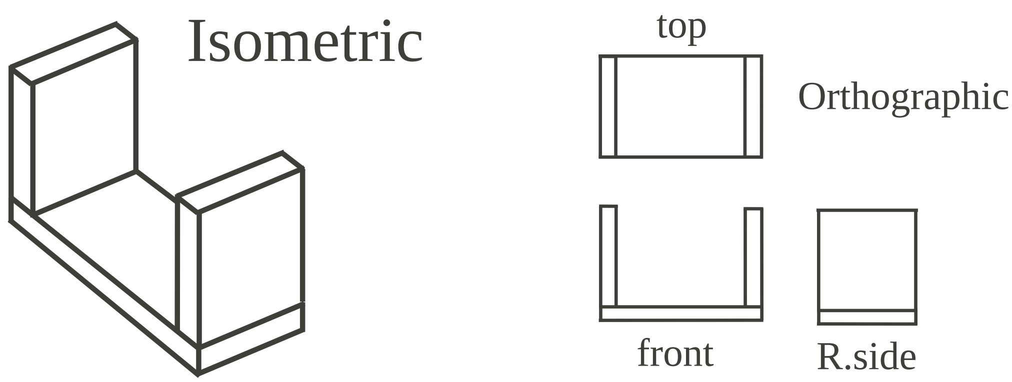
(b)



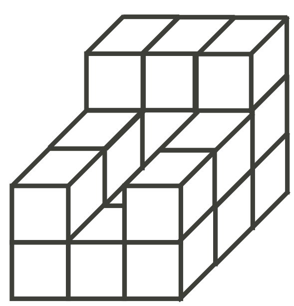
4. I take my square Double Bubble Gum wrapper (presumably with no crease before I fold) and fold it 4 times, creating an isosceles triangle each time. I want to create a rhombus in the middle of the unfolded paper so I can spit out my gum in there and wrap it. Does this fold technique create the rhombus I want?

5. A basic technical drawing of an object usually consists of two forms: **othographic** and **isometric**. An **orthographic drawing** of the object shows the object exactly as how one would see it from front, top and typically, right side view. An **isometric drawing** of the object is the object rendered in 3D where the front, top and right sides are all visible.

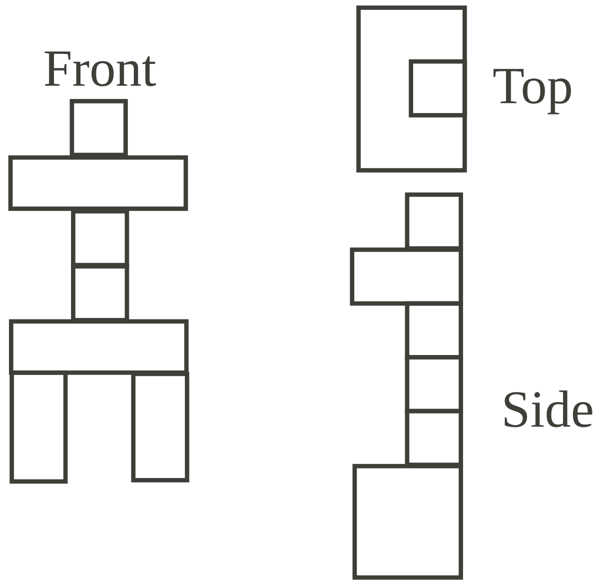
   For example, the images below are the isometric and orthographic drawings of the an upside down desk:



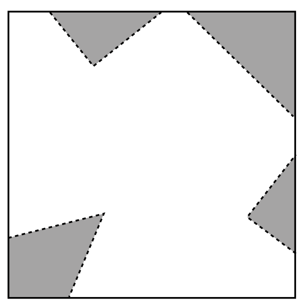
   (a) Below is my very comfortable chair made out of wooden blocks, in 3D! Draw the orthographic drawing of my chair.



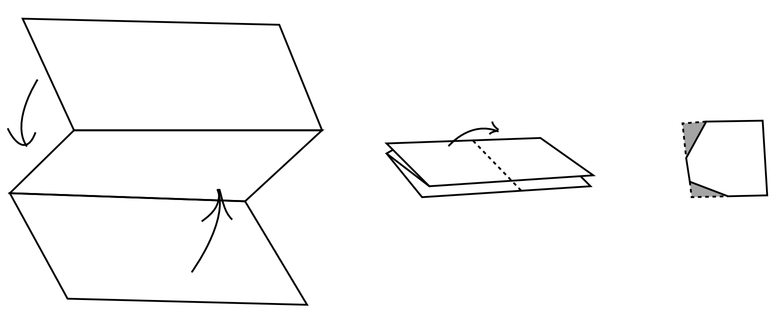
   (b) The Inukshuk is the symbolic rock of our gorgeous northern province Nunavut where you can gaze upon the aurora borealis in the crystal clear night sky. There, the Inuit people carry on their beautiful traditions and customs everyday. I made one out of wooden cube blocks and it looks like this in front, side and top view. Draw the isometric drawing of this Inukshuk.



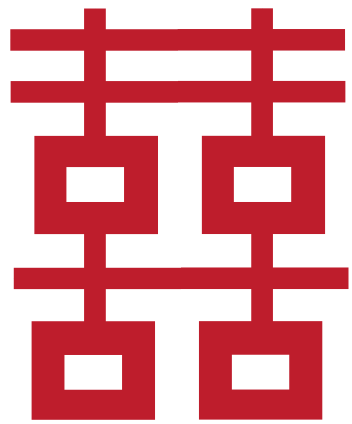
6. I fold my square paper in half twice such that I end up with a smaller square $\frac{1}{4}$ the original size. Then I cut out the gray regions, where the sharp quadrilateral is on the inner most corner of the fold. Unfold the paper, what does my unfolded paper look like?



7. I have a sheet of paper folded into 3 equal pieces, once over and once under. Then I fold in half again along the dotted line. Now I cut off the gray area, with the cut off area being part of the innermost edge. What does the paper look like after I unfold?



8. * The Chinese celebrates New Years using this special Chinese Character. It symbolizes happiness and prosperity.



   (a) Identify the number of times we fold a sheet of paper to get this character into its smallest component (i.e. no more lines of symmetry).

   (b) Can you find a way to cut this character from a folded sheet of paper?

9. ** **Let's Make Paper Snowflakes!** If you've made your fair share of paper snowflakes, you know it is easier to cut out paper snowflakes with 4 or 8 corners. But did you know that snowflakes actually have 6 corners? They do not actually have 4 or 8 corners and it is more difficult to cut out a 6 cornered snowflake. This is because it's much easier to continually fold the paper in half each time but much harder to trisect an angle. Try to cut a 6 cornered snowflake by folding the paper such that you get a regular hexagon after you unfold. You should still be cutting the smallest component only. (i.e. You only need to cut a pattern on the smallest component once to create 6 corners)