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Grade 7/8 Math Circles

February 7 & 8, 2017

Number Theory

Introduction

Today, we will be looking at some properties of numbers known as **number theory**. Number theory is part of a branch of mathematics called *pure mathematics*. More specifically, we will learn about palindromes and triangular numbers, before looking at prime numbers and some other pretty neat stuff.

Palindromic Numbers

What do you notice about the following images?



These images are symmetrical . However, it is not only pictures that can follow this property: words and numbers can as well.

Palindrome: A word or phrase that reads the same forwards and backwards. For example, Madam, Hannah, and Go dog are palindromes.

Palindromic Number: A number that reads the same forwards and backwards. For example, 1331, 404, 9, 77777, and 145686541 are palindromic numbers.

Examples At the Waterloo Marathon, everyone has a bib with a number on it. You are watching the runners going by and taking note of their bib number.

(a) James is the smallest 3-digit palindromic number. What number is James? 101

(b) The product of Maureen's two digits is 49. What palindromic number is Maureen? 77

Finding Palindromic Numbers

One way to find a palindrome is as follows:

1. Pick any number
2. Reverse the digits of the number
3. Add these two numbers together
4. Repeat until you get a palindrome

Example Using the number 37, find a palindromic number.

37 reversed is 73.
$37 + 73 = 110$.
110 reversed is 11.
$110 + 11 = 121$, which is a palindrome.

Perfect Square Palindromic Numbers

Evaluate the following:

$11^2 =$ 121

$101^2 =$ 10201

$1001^2 =$ 1002001

$10001^2 =$ 100020001

What is the pattern of these perfect squares?

The number of zeroes between (1,2) and (2,1) is the number of zeroes in the base.
(i.e. 1001 has two zeroes so $1001^2 = 1002001$.)

Examples

(a) What is $1\ 000\ 000\ 000\ 000\ 000\ 001^2$?

1 000 000 000 000 000 002 000 000 000 000 000 001

(b) What is $\sqrt{1\,000\,000\,002\,000\,000\,001}$?

1 000 000 001

Perfect Cube Palindromic Numbers

Let's see if we can find a similar pattern with perfect cubes.

$11^3 =$ 1331

$101^3 =$ 1030301

$1001^3 =$ 1003003001

$10001^3 =$ 1000300030001

What is the pattern of these perfect cubes?

The number of zeroes between (1,3), (3,3) and (3,1) is the number of zeroes in the base. (i.e. 1001 has two zeroes so $1001^3 = 1003003001$.)

Examples

(a) What is $1\ 000\ 001^3$?

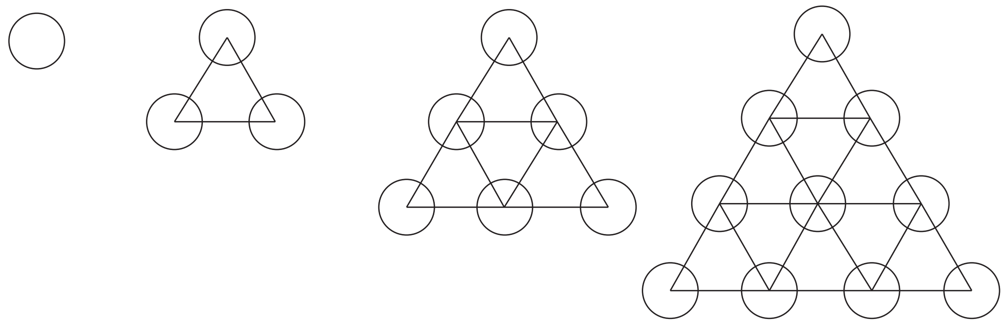
1 000 003 000 003 000 001

(b) What is $\sqrt[3]{1\,000\,000\,003\,000\,000\,003\,000\,000\,001}$?

1 000 000 001

Triangular Numbers

Consider the following pattern:



What is the rule of the pattern?

Add n dots to the $(n-1)^{th}$ triangle.

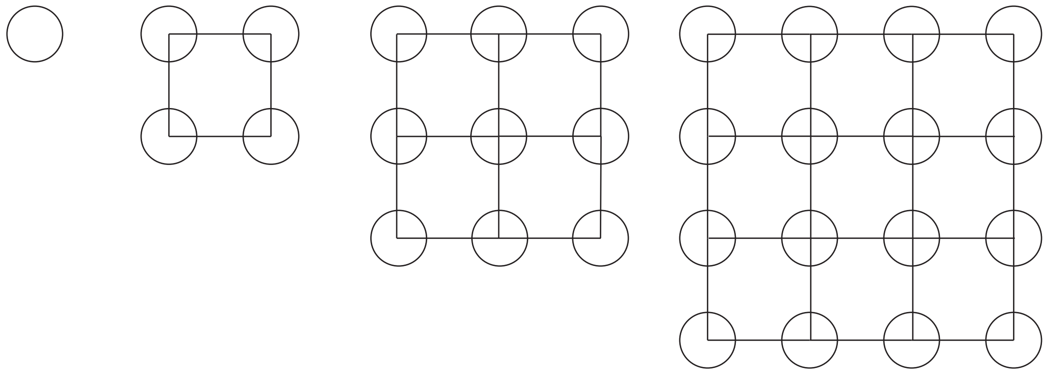
The number of dots in each triangle form a sequence of numbers that we call the **triangular number sequence**. The sequence of triangular numbers is as follows: {1,3,6,10,...}. We also see this sequence in Pascal's triangle!

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

From this sequence, we can find that the formula for triangular numbers is $t_n = \dfrac{n(n+1)}{2}$, where t_n is the n^{th} term or the n^{th} triangle.

But triangles aren't the only shape we can consider. In fact, we can consider any shape, so we could have any set of **polygonal numbers**.

Let's consider the set of square numbers:



Notice that these numbers look familiar! We tend to know the square numbers since we know our square roots so well. We also call the set of square numbers **perfect squares** since their square roots are integers.

Examples

(a) What is the 13^{th} triangular number?

$$\mathrm{t}_{13} = \frac{13 \times (13+1)}{2} = \frac{13 \times 14}{2} = 91$$

(b) What is the 13^{th} square number?

$$\mathrm{t}_{13} = 13^2 = 169$$

The Locker Problem

One hundred students are assigned lockers 1 to 100. The student assigned to locker 1 opens every locker. The student assigned to locker 2 then closes every other locker. The student assigned to locker 3 changes the status of all lockers whose numbers are multiples of 3 (If a locker that is a multiple of 3 is open, the student closes it. If it is closed, the student opens it). The student assigned to locker 4 changes the status of all lockers whose numbers are multiples of 4, and so on for all 100 lockers.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

You may use the grid above to help solve the following questions:

1. Which lockers are left open? And why were they left open?

   Lockers 1, 4, 9, 16, 25, 36, 49, 64, 81, 100 were left open because they are perfect squares. Perfect squares have an odd number of factors. For example, 25 has factors 1, 5 and 25. Locker 25 was opened by the 1^{st} student, closed by the 5^{th} student and opened again by the 25^{th} student.

2. Which lockers were touched exactly two times?

   2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97

3. How do you know that these lockers were touched exactly two times?

   Because they are all prime numbers! They have exactly two factors.

Prime Numbers

A **prime** number is a natural number that can only be divided by 1 and itself.

A **composite** number is a natural number that has more **factors** than just 1 and itself.

For example, 2, 3, and 5 are prime numbers, since $2 = 2 \times 1$ but there is no other way to multiply to get 2. 4, 6, and 9 are composite numbers since $4 = 2 \times 2$ and $4 = 4 \times 1$.

One method for finding the prime numbers is by using the **Sieve of Eratosthenes**. Here are the steps to this algorithm, using the following table:

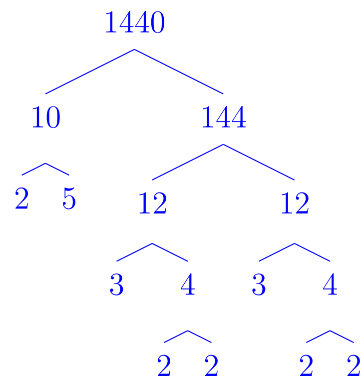
1. Cross out 1 (it is not prime)
2. Circle 2 (it is prime) and then cross out all multiples of 2
3. Circle 3 (it is prime) and then cross out all multiples of 3
4. Circle 5, then cross out all multiples of 5
5. Circle 7, then cross out all multiples of 7
6. Continue by circling the next number not crossed out, then cross out all of its multiples

The circled numbers are all the prime numbers less than 100.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Prime Factorization: Extended

Review: Find the prime factorization of 1440. Factor trees may vary.



$$1440 = 2^5 \times 3^2 \times 5$$

Fermat's Factorization Method

Find the prime factorization of 989.

$$989 = 23 \times 43$$

We know that every positive number can be written as a product of prime factors and this product can be found using prime factorization. What happens if the prime factorization is composed of large prime numbers? It becomes difficult and to manually check if each prime number is a factor until one is found. Another method to help us find the prime factorization of such a number is something called **Fermat's Factorization Method**. Before we begin, we need to learn one new thing called the **difference of squares**.

> **Difference of Squares**
>
> Let a and b any number. Then,
>
> $$a^2 - b^2 = (a+b)(a-b)$$

Examples Evaluate the following:

(a) $5^2 - 4^2 = (5+4)(5-4) = 9 \times 1$

(b) $12^2 - 5^2 = (12+5)(12-5) = 17 \times 7$

Now, to find the prime factorization of a number using Fermat's Factorization Method, we will use the following steps.
Suppose we want to find the prime factorization of the number n.

1. Choose the smallest number a such that $a^2 > n$.
2. Evaluate $a^2 - n$. If $a^2 - n$ is NOT a perfect square, then repeat step 2 with $(a+1)$, $(a+2), (a+3), \ldots$ until we find a perfect square.
3. Suppose a gives us a perfect square, b^2, in step 2. Then,

$$a^2 - n = b^2 \Rightarrow n = a^2 - b^2 = (a+b)(a-b)$$

4. If $(a+b)$ and $(a-b)$ are prime numbers, then we are done. Otherwise, we can do one of two things:
   (a) Repeat Fermat's Factorization Method with $(a+b)$ and/or $(a-b)$, OR
   (b) Use a factor tree to find the prime factorizations of $(a+b)$ and/or $(a-b)$.

Example Find the prime factorization of 1173.

The smallest number a such that $a^2 > 1173$ is 35.

a	$a^2 - 1173$	$b = \sqrt{a^2 - 1173}$	$a+b$	$a-b$
35	52	-	-	-
36	123	-	-	-
37	196	14	51	23

So, $1173 = 51 \times 23$. 23 is prime but 51 is not so we still need to find the prime factorization of 51. We can use Fermat's Factorization Method again or use a factor tree.

$$\begin{array}{c} 51 \\ \swarrow \quad \searrow \\ 3 \qquad 17 \end{array}$$

$$51 = 3 \times 17 \Rightarrow 1173 = 51 \times 23 = (3 \times 17) \times 23$$

Therefore, the prime factorization of 1173 is $\underline{1173 = 3 \times 17 \times 23}$

Multiplicity of Prime Factors

Multiplicity: The number of times a prime factor is multiplied. For example, $9 = 3^2$. The multiplicity of 3 is 2.

Find the prime factorization of the following perfect squares:

(a) 25 $= 5^2$ (b) 81 $= 3^4$ (c) 144 $= 3^2 \times 2^4$

What do you notice about the multiplicity of each prime factor in the examples above?

The multiplicity of each prime factor is even or are multiples of 2.

Interesting. Let's try to find the multiplicity of a few perfect cubes.

(a) 8 $= 2^3$ (b) 64 $= 2^6$ (c) 8000 $= 2^6 \times 5^3$

What do you notice about the multiplicity of each prime factor in the examples above?

The multiplicity of each prime factor is a multiple of 3.

Therefore, the multiplicity of each prime factor of an n^{th} root is a multiple of n.

Examples

(a) If you know that 243 is a 5^{th} root, find its prime factorization.

Start with the smallet prime number. Is 2^5 a possible factor of 243? No, because 2 does not divide 243. Try the next prime number. Is 3^5 a factor of 243? Yes, in fact $3^5 = 243$. Thus, $243 = 3^5$

(b) What is the multiplicity of the prime factors of 2187?

$2187 = 3^7$. The multiplicity of 3 is 7.

How Many Factors?

Without listing all the factors, how many factors does 1440 have?

We can answer this question using prime factorization! Consider this:

What is the prime factorization for each of the following numbers? List the factors for each number as well.

8 $= 2^3 \Rightarrow \{\ 1,\ 2,\ 4,\ 8\ \}$

9 $= 3^2 \Rightarrow \{\ 1,\ 3,\ 9\ \}$

16 $= 2^4 \Rightarrow \{\ 1,\ 2,\ 4,\ 8,\ 16\ \}$

There are 4, 3 and 5 factors respectively.

What do you notice?

The number of factors of a number is the multiplicity of its prime factor plus one!

What is the prime factorization of 24? List all its factors.

$24 = 2^3 \times 3 \Rightarrow \{\ 1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 12,\ 24\ \}$
2^3 has 4 factors and 3^1 has 2 factors and so there are $4 \times 2 = 8$ factors of 24.

Number of Factors of N

Let N be any positive integer. Suppose the prime factorization of N is

$$N = 2^a \times 3^b \times 5^c \times \ldots$$

where a, b and c are also positive integers. Then, the number of factors of N is

$$(a+1) \times (b+1) \times (c+1) \times \ldots$$

More Fun With Primes!

In keeping with the topic of palindromes, a **palindromic prime** is a prime number that reads the same forwards and backwards. Here is a list of some palindromic primes:

2, 3, 5, 7, 11, 101, 131, 151, 181, 191, 313, 353, ...

Twin primes are pairs of primes numbers that are either two less or two more than each other. For example, (5,7), (17,19) and (41,43) are pairs of twin primes.

An **emirp** is a prime that gives you a *different* prime when its digits are reversed. For example, `13` becomes `31` when we reverse its digits and they are different numbers and they are both prime. A list of some emirp numbers are shown below:

13, 17, 31, 37, 71, 73, 79, 97, 107, 113, 149, 157, ...

A **Mersenne prime** is a prime number that is one less than a power of two. To be more clear, suppose `p` can by any whole number. Mersenne primes can be written as follows:

$$2^{p} - 1$$

The first three Mersenne primes are:

$$\begin{array}{ccc} 3, & 7, & 31 \\ \Downarrow & \Downarrow & \Downarrow \\ 2^2 - 1 & 2^3 - 1 & 2^5 - 1 \end{array}$$

Fun fact!
Currently, the largest known Mersenne prime is $2^{74207281} - 1$. It is 22 338 618 digits long!

Problem Set Solutions

1. How many 1-digit palindromic numbers are there?
   There are 9 1-digit palindromic numbers. (1, 2, 3, 4, 5, 6, 7, 8, 9)

2. Which of the following are palindromes?
   (a) HANNAHBANANABHANNAH Palindrome
   (b) 1030201 Not a palindromic number
   (c) ABCDECBA Not a palindrome
   (d) 1771 Palindromic number

3. Find a palindromic number from the following:
   (a) 18 $\Rightarrow 18 + 81 = 99$
   (b) 886

$$\begin{aligned}
&\Rightarrow \quad 886 + 688 = 1574 \\
&\Rightarrow \quad 1574 + 4751 = 6325 \\
&\Rightarrow \quad 6325 + 5236 = 11561 \\
&\Rightarrow \quad 11561 + 16511 = 28072 \\
&\Rightarrow \quad 28072 + 27082 = 55154 \\
&\Rightarrow \quad 551554 + 455155 = 100309 \\
&\Rightarrow \quad 100309 + 903001 = 1003310 \\
&\Rightarrow \quad 1003310 + 133001 = 1136311
\end{aligned}$$

   (c) 3742 $\Rightarrow 3742 + 2473 = 6215 \Rightarrow 6215 + 5126 = 11341 \Rightarrow 11341 + 14311 = 25652$

4. 2002 was the last palindromic year. What will be the next palindromic year? 2112

5. How many 7-digit palindromic numbers are there?

$$\underline{9} \times \underline{10} \times \underline{10} \times \underline{10} \times \underline{1} \times \underline{1} \times \underline{1} = 9 \times 10^3 \times 1^3 = 9000$$

   There are 9 possible options for the 1^{st} digit (exclude 0 since a number cannot begin with 0). Since we are looking for palindromic numbers, the last digit must be the same as the 1^{st} digit so there is 1 possible option for the last digit. Next, there are 10 options for the 2^{nd} digits and 1 option for the 6^{th} digit, then 10 options for the 3^{rd} digits and 1 option for the 5^{th} digits, and finally, 10 options for the 4^{th} digit. Therefore, there are $9 \times 10 \times 10 \times 10 \times 1 \times 1 \times 1 =$ 9000 different 7-digit palindromic numbers.

6. How many palindromic numbers are there between the numbers 100 and 400?

   Since the palindromic numbers must be between 100 and 400, we are looking for 3-digit palindromic numbers. For the 1^{st} digit, the only possible numbers are 1, 2 and 3 (if 4 was an option then the number would be 4 _ 4, but 4 _ 4 is larger than 400 so it cannot be counted). There are 3 possibilities for the 1^{st} digit and thus, there is 1 possibility for the 3^{rd} digit. There are 10 possibilities for the 2^{nd} digit. Therefore, there are $3 \times 10 \times 1 =$ 30 different palindromic numbers between 100 and 400.

7. Maria is finishing a marathon and Paul is waiting for her at the finish line. He cannot remember her bib number, just that it was the next palindromic number after his bib number, 5678. What bib number does Maria have?

   Let's first check if there exists a palindromic number in the 5000s that is greater than Paul's bib number, 5678. Assuming 5 is the first digit of Maria's bib number, then the last digit must be 5 as well. So far, Maria's bib number is 5 _ _ 5. Notice the second digit must be greater than 6. Otherwise, Maria's bib number would be 5665 but 5665 is less than 5678. The next digit to try is 7. If the second digit is 7, then Maria's bib number is 5775 and 5775 is a palindromic number greater than 5678. Therefore, Maria's bib number is 5775.

8. (a) Evaluate the following:

      i. $\sqrt{100\,000\,020\,000\,001}$ $= 10\ 000\ 001$

      ii. $100\ 000\ 001^3$ $= 1\ 000\ 000\ 030\ 000\ 000\ 300\ 000\ 001$

   (b) (Square of Perfect Fourths) $101^4 = 104060401$. Evaluate the following:

      i. $\sqrt[4]{10\,004\,000\,600\,040\,001}$ $= 1\ 0001$

      ii. 100001^4 $= 100\ 004\ 000\ 060\ 000\ 400\ 001$

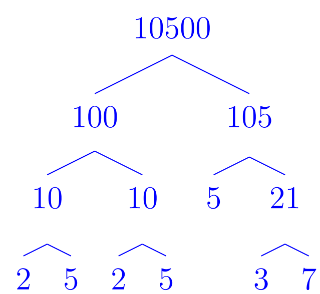
   (c) What do you notice about the non-zero digits of the perfect square, cube, and fourth palindromic numbers?

      The non-zero digits of the palindromic numbers of the n^{th} root are the numbers of the n^{th} row of Pascal's Triangle!

9. Sachin plays for the Waterloo quidditch team and made sure he picked out a jersey number that is a palindrome. Quidditch rules state that his jersey number must be less than 1000. Sachin chose an even number, where the product of the digits in his number is 12. What is Sachin's jersey number?

   First, the single-digit factors of 12 are 1, 2, 3, 4, and 6 (12 is a factor but its digits 1, 2 are factors of 12 as well so we can exclude 12). 0 cannot be a digit of Sachin's jersey number because then the product of the digits is 0. Since the first and last digit are the same and the jersey number is even, the first and last digit must be even. If 6 is the first and last digit, then the number must be $6n6$ where n is any possible digit. However, $6 \times n \times 6 = 36n > 12$. 6 is not the first and last digit. If 4 is the first and last digit, then the number must be $4n4$. However, $4 \times n \times 4 = 16n > 12$. 4 is also not the first and last digit. This means 2 must be the first and last digit. It works because the number must be $2n2$ and $2 \times n \times 2 = 4n$ and $4n = 4 \times 3 = 12$. Since $12 = 2 \times n \times 2 = 2 \times 3 \times 2$, 3 must be the second digit. Therefore, Sachin's jersey number is <u>232</u>.

10. (a) Find the prime factorization of 10 500. Factor trees may vary.



$$10\,500 = 2^2 \times 3 \times 5^3 \times 7$$

   (b) How many factors does 10 500 have?

   There are $(2+1) \times (1+1) \times (3+1) \times (1+1) = 3 \times 2 \times 4 \times 2 = 48$ factors.

11. Given the expression $2^{2k}\, 3^{3k}\, 5^{5k} = 337\,500$, what is k? $k = 1$

12. Find the prime factorization of the following numbers.

(a) 4897

We can use Fermat's Factorization Method.

The smallest number a such that $a^2 > 4897$ is 70.

a	$a^2 - 4897$	$b = \sqrt{a^2 - 4897}$	$a+b$	$a-b$
70	3	-	-	-
71	144	12	83	59

Since 83 and 59 are prime, the prime factorization of 4897 is $\underline{4897 = 59 \times 83}$.

(b) 1219

We can use Fermat's Factorization Method.

The smallest number a such that $a^2 > 1219$ is 35.

a	$a^2 - 1219$	$b = \sqrt{a^2 - 1219}$	$a+b$	$a-b$
35	6	-	-	-
36	77	-	-	-
37	150	-	-	-
38	225	15	53	23

Since 53 and 23 are prime, the prime factorization of 1219 is $\underline{1219 = 53 \times 23}$.

(c) 4085

We can use Fermat's Factorization Method.

The smallest number a such that $a^2 > 4085$ is 64.

a	$a^2 - 4085$	$b = \sqrt{a^2 - 4085}$	$a+b$	$a-b$
64	11	-	-	-
65	140	-	-	-
66	271	-	-	-
67	404	-	-	-
68	539	-	-	-
69	676	26	95	43

So far, we have that $4085 = 95 \times 43$. 43 is prime but 95 is not. Using a factor tree, we see that $95 = 5 \times 19$.

$$\begin{array}{c} 95 \\ \diagup \quad \diagdown \\ 5 \quad 19 \end{array}$$

$$4085 = 95 \times 43 = (5 \times 19) \times 43$$

Therefore, the prime factorization of 4085 is $\underline{4085 = 5 \times 19 \times 43}$.

13. What is the multiplicity of the prime factors of 9 765 625?

    The prime facotorization is 9 765 625 $= 5^9$. The multiplicity of 5 is 9.

14. Is 129 a Mersenne prime? No, the closest Mersenne prime to 129 is $127 = 2^7 - 1$.

15. * **Locker Problem Extended** Which locker(s), from 1 to 100, were open and closed the most times?

16. ** An old cat lady meets with her fellow animal loving friend. The cat lady asks her friend, "How many dogs do you have?" The friend answers, "I have three dog whom I love very much." The cat lady then asks, "How old are your dogs?" The friend decides to test the cat lady's math skills and replies, "The product of their ages 288 and the sum of their ages is the day of month on which your birthday falls." After some thinking, the cat lady says, "Nope, I need more information." "Alright," says the friend, "The youngest dog still needs to be potty-trained." Immediately, the cat lady says, "Got it! I know their ages!!" How old are the dogs?

    Make a chart listing the possible ages of the dogs and the sum of the ages given that the product of the ages must be 288:

Dog 1	**Dog 2**	**Dog 3**	**Sum of Ages**	**Dog 1**	**Dog 2**	**Dog 3**	**Sum of Ages**
1	1	288	290	2	6	24	32
1	2	144	147	2	8	18	28
1	3	96	100	2	9	16	27
1	4	72	77	2	12	12	26
1	6	48	55	3	3	32	38
1	8	36	45	3	4	24	31
1	9	32	42	3	6	16	25
1	12	24	37	3	8	12	23
1	16	18	35	4	4	18	26
2	2	72	76	4	6	12	22
2	3	48	53	4	8	9	21
2	4	36	42	6	6	8	20

    Since the cat lady could not solve the question with this much information, we can say that the sum of the dogs' ages is 26 (since there are two possibilities with a sum of 26 and $26 \leq 31$ days of a month). The friend mentioned that the *youngest* dog still needs to be potty-trained which means there can only be one youngest dog. Therefore, the ages of the three dogs are 2, 12 and 12.

17. *** How integers n are there where $1 \leq n \leq 100$ and n^n is a perfect square?

A perfect square can be written as x^2. We notice that all n^n where n is even is a perfect square. For example, $4^4 = 4 \times 4 \times 4 \times 4 = 4^2 \times 4^2 = (4^2)^2$ is a perfect square. There are 50 even integers from 1 to 100. Thus, we have counted 50 integers so far. Next, 1^1 is a perfect square but 1 is odd. Check if there are any other odd numbers that satisfy our criteria. We find that 9 works.

$$9^9 = 9 \times 9 \times 9 \times 9 \times 9 \times 9 \times 9 \times 9 \times 9 = 9^4 \times 9^4 \times 9 = (9^4)^2 \times 9 = (9^4)^2 \times 3^2$$

All the odd perfect squares work as well. So we add 1, 9, 25, 49 and 81 to our count. Therefore, there are $\underline{50 + 5 = 55}$ integers from 1 to 100 that are perfect squares when raised to the power of itself.