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Centre for Education in
Mathematics and Computing

Grade 6 Math Circles

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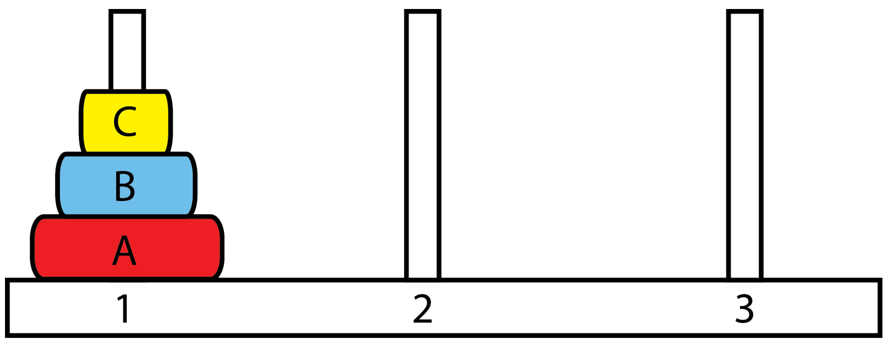
Patterns

Tower of Hanoi



The Tower of Hanoi is a puzzle with three wooden pegs and three rings. These rings are each different sizes and the puzzle begins with the rings all on the first peg, stacked with each ring smaller than the one below it.

The goal of this puzzle is to move the rings to another peg in the minimum number of moves. However, you can only move the rings one at a time and you may never place a larger ring on top of a smaller one!

What is the minimum number of moves to solve the puzzle?

What if you now have 4 rings to move? What is the minimum number of moves to solve the puzzle now?

What about with 5 rings?

What is happening every time we add another ring to the puzzle? Predict how many moves you will need to solve the puzzle with 6, 7, and 8 rings.

Patterns

A pattern is a set of numbers or objects that are related to each other by a specific **rule**. The rule explains how a pattern starts and what comes next. If you were a told only the rule, you should be able to replicate the pattern exactly. Each object in a pattern is called a **term**, and each term has a specific place in the pattern.

Patterns show up not only on your math homework, but all over the place in the real world! The music we listen to is often made of repeated notes and chords; nature is filled with organic patterns; and geometric patterns can be found in magazines, on clothes, and decorating your home. Being able to recognize patterns is a key to mathematical thinking!

Example 1

$$\leftarrow \quad \downarrow \quad \rightarrow \quad \downarrow \quad \leftarrow \quad \downarrow \quad \rightarrow \quad \downarrow \quad \leftarrow \quad \ldots$$

Note: "..." in patterns translates to "and so on."

Draw the next four terms in this pattern.

What is the rule for this pattern?

What is the 99th term?

Example 2

The seven dwarfs are playing a countdown game on their way to work. Starting at the front of the line, they each count a number starting from 100 down to 1. When they reach the end of the line, they loop back to Doc at the front. The order they are walking is Doc, Grumpy, Happy, Sleepy, Sneezy, Bashful, and Dopey.

Who says 1?

Sequences

The two examples we have looked at so far both repeat a set over and over again. These types of patterns are often easy to recognize. Let's look at another type of pattern. **Sequences** are patterns where the rule relates each term to the previous term.

An example of a sequence would be: 23, 78, 133, 188, 243, ...

We can see that the rule this sequence is that, starting from 23, every term is 55 more than the previous term. An easy way to recognize patterns in sequences is writing down the differences between terms:

	1st Term	2nd Term	3rd Term	4th Term	5th Term
Term	23	78	133	188	243
Difference		55	55	55	55

You may not always need to use this method to find the rule of a sequence, but it can be helpful with a tricky problem!

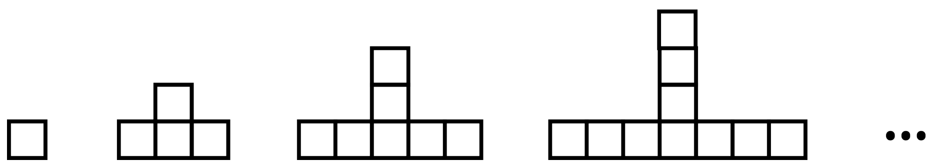
Example 3

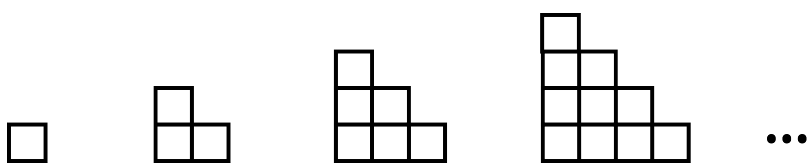
What is the rule for the following patterns:

1. 5, 11, 17, 23, 29, ...

2. 5, 6, 10, 17, 27, 40, 56, ...

3. 2, 4, 8, 16, 32, 64, 128, ...

4. 

5. 

Example 4

Lumiere, Cogsworth, Mrs. Potts, Babette and Chip are planting a line of flowers together.

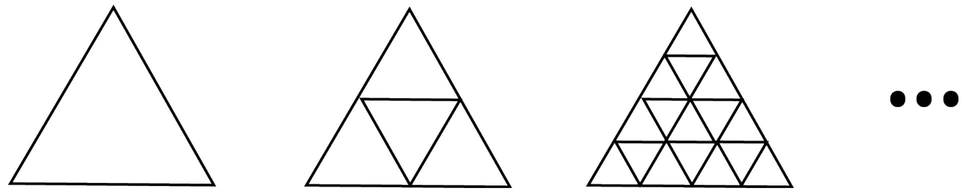
- Lumiere plants five primroses in a line
- Cogsworth plants petunias in each of the spaces between the primroses
- Mrs. Potts plants peonies in the spaces between the flowers already in line
- Babette plants poppies in the spaces between the flowers already in line
- Finally, Chip plants pansies in the spaces between the flowers already in line

How many flowers are in a row after each person's turn to plant?

Drawing a diagram might be helpful to understand the pattern.

Example 5

Take a look at the following pattern.



1. Describe the rule for this pattern.

2. Draw the next term of this pattern.

3. Count the number of smallest triangles (triangles which do not have smaller triangles within) in each term. What is the rule for this number pattern?

4. Now count the number of total triangles of any size in each term. What is the rule for this number pattern? (Only go up to the 3rd term)

Looking at patterns on their own may be fun and interesting, but did you know that patterns can be used to solve arithmetic problems as well?

Sum of Consecutive Numbers

Let's look at the pattern we get when we take the sum of consecutive numbers:

$1 + 2 + 3 =$ $\quad$ $1 + 2 + 3 + 4 =$ $\quad$ $1 + 2 + 3 + 4 + 5 =$

$1 + 2 + 3 + 4 + 5 + 6 =$ $\quad$ $1 + 2 + 3 + 4 + 5 + 6 + 7 =$

Do you notice a pattern developing? In fact, we have already seen patterns like this!

	1st Term	2nd Term	3rd Term	4th Term	5th Term
Term					
Difference					
Difference					

We have found the pattern for the sum of consecutive numbers. But what if I asked you for the sum of 1 to 100? That would be the 100th term of this pattern! Even knowing the rule for this pattern, finding the 100th term would take some time. But the famous mathematician **Carl Gauss**, at only age 10, figured out the answer in just a few moments! He found another pattern in the numbers to help him solve this problem.

Gauss noticed that the numbers from 1 to 100 were made up of **pairs that sum to 101**:

$$1 + 2 + 3 + 4 + \ldots + 97 + 98 + 99 + 100$$

$1 + 100 = 101$ $\quad$ $2 + 99 = 101$ $\quad$ $3 + 98 = 101$

From there, he reasoned that dividing the 100 numbers into pairs results in $100 \div 2 = 50$ pairs. Now, he knows the sum of 1 to 100 is $101 \times 100 \div 2 = 5050$.

It turns out this rule can be generalized to find the sum of 1 to n for any positive whole number n. Let $S_n = 1 + 2 + 3 + ... + (n-2) + (n-1) + n$ represent this sum. Using Gauss' method, we can find S_n with the formula:

$$S_n = (n+1) \times n \div 2$$

Let's use this formula to check that $S_7 = 1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$

Not only can patterns be surprisingly helpful in math problems, but they show up in unexpected places in the real world!

Fibonacci and the Golden Ratio

If you were here for the Math Circle about Pascal's Triangle in the Fall, you might remember the following sequence: 1, 1, 2, 3, 5, 8, 13, ...

This is called the **Fibonacci Sequence**. What is the rule for this sequence?

Leonardo Pisano, better known as Fibonacci, was an Italian mathematician who brought the decimal number system to Europe and replaced Roman Numerals. As you might expect, he also discovered the **Fibonacci sequence**, the most famous mathematical sequence in the world.

But why is this sequence so famous?

Take a look at what happens when you divide pairs of consecutive terms:

$1 \div 1 =$ $\qquad$ $2 \div 1 =$

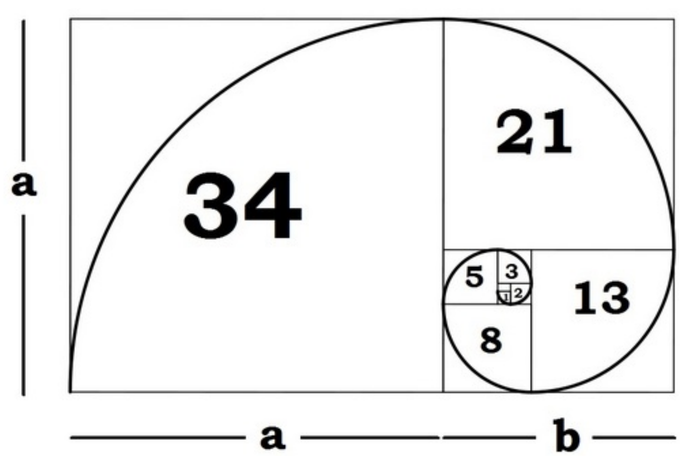
$3 \div 2 =$ $\qquad$ $5 \div 3 =$

$8 \div 5 =$ $\qquad$ $13 \div 8 =$

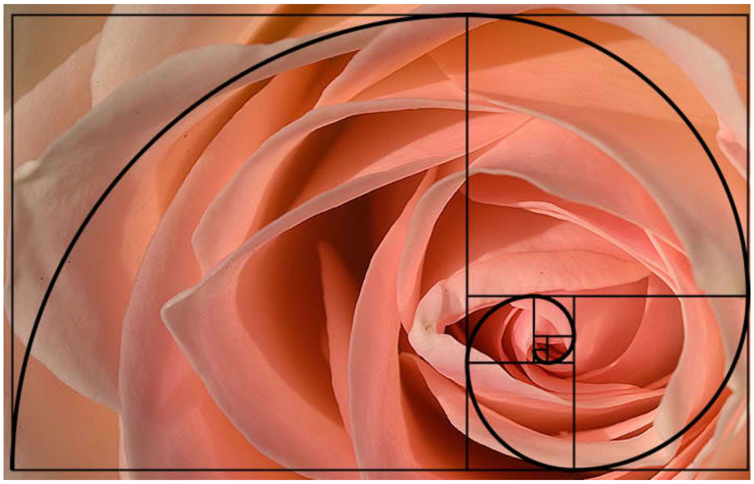
$21 \div 13 =$ $\qquad$ $34 \div 21 =$

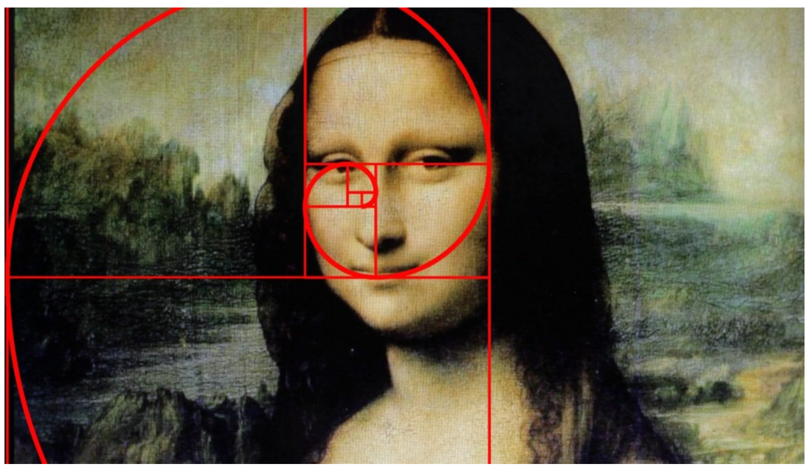
Do you see a trend? It turns out, as you keep diving terms further on in the Fibonacci Sequence, you approach a number known as the **Golden Ratio**. This number is actually equal to $\frac{1+\sqrt{5}}{2} = 1.618033989...$, a decimal that goes on forever! Why is this number called the Golden Ratio? It turns out that the Fibonacci Sequence and Golden Ratio has been long thought to have universal visual appeal. It appears in flower petals, the way that seeds grow, tree branches, shells, the galaxy, architecture, art, DNA molecules, and even our faces!

The Golden Ratio's visual representation is a spiral that results from drawing a curve through squares with the Fibonacci numbers as the side lengths.



Here are some examples of the Golden Ratio appearing in the real world:





Problem Set

1. Give an example of a pattern in your daily life.

2. For each of the following patterns, describe the rule and write the next two terms in the pattern.

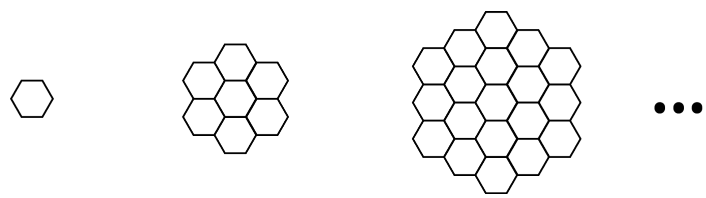
   (a) 3, 2, 1, 2, 1, 1, 3, 2, 1, 2, 1, 1, 3, 2, ...

   (b) 3, 6, 12, 24, 48, 96, ...

   (c) 1, 2, 6, 24, 120, 720, ...

   (d) △ ● □ ▲ ○ △ ● □ ▲ ○ △ ● □ ...

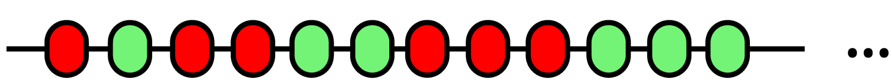
   (e) Instead of drawing the next two terms, you can list the number of small hexagons in the next two terms.



3. Moana voyages to a different island every week in the order: Tetiaroa, Fiji, Bora Bora, Tetiaroa, Fiji, Bora Bora, ...

   If on the first week of the year Moana goes to Tetiaroa, where is she at the end of the year (on the 52nd week)?

4. Beads are placed on a string in the following pattern: 1 red, 1 green, 2 red, 2 green, 3 red, 3 green, ... with the number of each colour bead increasing by one every time a new group of beads is placed. How many of the first 100 beads are red?



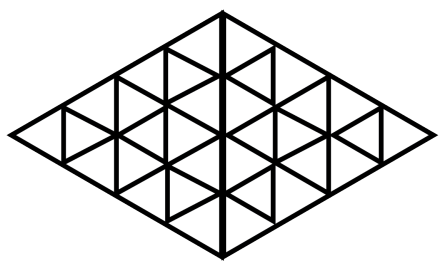
5. Olaf has a cold and sneezes out tiny snowmen who can multiply! Each 10 minute time period, one quarter of the tiny snowmen melt but the remaining all double! If Olaf sneezes out 32 snowmen initially, how many are there after 40 minutes?

6. Positive numbers are arranged in a grid with 5 columns (a, b, c, d, and e) in the following pattern:

a	**b**	**c**	**d**	**e**
1	2	3		
	4	5	6	
		7	8	9
10	11	12		
	13	14	15	
		16	17	18

If the pattern continued, which column would 111 be in?

7. All the small triangles in this diagram are equilateral triangles. What is the total number of equilateral triangles of any size? Use techniques we discussed in Math Circles to solve this problem.



8. A piece of paper that is 0.4mm thick is folded in half. It is then folded in half 5 more times! What is the final thickness of the folded paper?

9. Take a look at the following products:

$$999 \times 222\ 222 = 221\ 999\ 778$$
$$999 \times 333\ 333 = 332\ 999\ 667$$
$$999 \times 444\ 444 = 443\ 999\ 556$$

Notice a pattern to the answers? Use this pattern to find the answer to $999 \times 777\ 777$.

10. What is the sum of all odd numbers from 1 to 1000?

11. Suppose you put two new born rabbits (one male and one female) together to study their breeding habits. Rabbits breed according to the following rules:

    - Each pair of rabbits breed a pair of rabbits (one male and one female) each month starting from when they are 2 months old.
    - Each pair of these newly bred rabbits also start breeding pairs of rabbits each month from when they are 2 months old.

    (a) How many pairs of rabbits are there after 0, 1, 2, 3, and 4 months? Drawing a tree diagram may be helpful.

    (b) Does this pattern look familiar? What is the rule for this pattern?

    (c) How many pairs of rabbits are there after a year (at Month 12)?

12. **Challenge problem.** The following pattern is called the "Look and Say" sequence: 1, 11, 21, 1211, 111221, 312211, 13112221, ...

    (a) What is the rule for this pattern?

    (b) Does the digit "4" ever appear in this pattern?

13. **Challenge problem.** Follow these instructions to create the Koch Snowflake:

    1. Draw a large equilateral triangle on a sheet of paper with pencil.
    2. Divide each line segment of triangle into three equal sections.
    3. Draw two lines the same length as the middle section, to form an equilateral triangle pointing outwards, with the middle section of the three you created in the previous step as the base.
    4. Erase the middle section, leaving the triangle you just drew with only two sides.
    5. Repeat the above steps for each of the new smaller triangles you have just created. Repeat two more times.

    These steps could theoretically be repeated infinitely (if only you could draw infinitesimally small triangles)! This type of repeated visual pattern is called a fractal.