Intermediate Math Circles  
Wednesday February 24, 2016  
Introduction to Vectors

We write vectors using vector notation. This includes

- Write components as column with square brackets.
- Using a directed line segment (ray).
- Use an arrow over letter to tell its a vector.

\[ \vec{z} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \]

Definition 1: A vector must have
- Direction.
- Length.

Definition 2: In 2D, the zero vector has both components equal to 0. It is denoted as \( \vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) and has no length and an undefined direction.

Vector Addition
We can add vectors by adding their components.

Example:
\[
\begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 + 0 \\ 4 + 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}
\]
\[
\begin{bmatrix} 1/2 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 + 3 \\ 2 + 1 \end{bmatrix} = \begin{bmatrix} 7/2 \\ 3 \end{bmatrix}
\]

Graphing Addition of Vectors

Parallelogram Rule  
"Tip to Tail"
Scalar Multiplication

**Definition 3:** If \( \vec{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \) and \( t \) is a scalar, then scalar multiplication is defined as

\[
 t\vec{z} = t \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} tz_1 \\ tz_2 \end{bmatrix}
\]

Example:

\[
 2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 2 \times \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \times 2 \\ 2 \times 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}
\]

Subtraction is a combination of both addition and scalar multiplication.

**Graphing Scalar Multiplication**

Scalar multiplication _stretches_ or _shrinks_ a vector.

Vectors and Lines

Consider the vector \( \vec{z} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \).

- How can we turn \( \vec{z} \) into a line passing through the origin?

\[
 \vec{z} = t \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \ t \in \mathbb{R}
\]

- How can we make this line pass through the point \((1, 1)\)?

\[
 \vec{z} = t \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]
Definition 4: A line through $\vec{p}$ with direction vector $\vec{d}$ has the vector equation
\[ \vec{z} = \vec{p} + t\vec{d}, \quad t \in \mathbb{R} \]

Question: How can we tell if two vector equations are parallel?

Two vectors are parallel if their direction vectors, $\vec{d}$) are scalar multiples of each other.

Definition 5: The parametric equation of the line $\vec{z} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + t \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ is the collection of equations
\[ z_1 = p_1 + td_1 \]
\[ z_2 = p_2 + td_2 \]

Example: Does the vector equation $\vec{z} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ pass through the point (0, -1)?

Putting $\vec{z}$ into parametric form:
\[ z_1 = -2 + t(-1) \quad \Rightarrow z_1 = -2 - t \]
\[ z_2 = 1 + t(1) \quad \Rightarrow z_2 = 1 + t \]

We want $z_1 = 0$, therefore $0 = -2 - t$ or $t = -2$.
We check this in our second equation:
\[ z_2 = 1 + (-2) = -1 \]
as required.

For a parametric equation, the value of $t$ must be the same for every equation!

Example: Solve for the unknown variables.
\[ m \begin{bmatrix} 1/2 \\ 3 \end{bmatrix} + n \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 9 \end{bmatrix} \]

We split this into its parametric equations:
\[ m(1/2) + n(-3) = 11 \quad \Rightarrow \frac{m}{2} - 3n = 11 \quad (1) \]
\[ m(3) + n(1) = 9 \quad \Rightarrow 3m + n = 9 \quad (2) \]

Isolate for $n$ in equation (2): $n = 9 - 3m$

Substitute this into equation (1): $\frac{m}{2} - 3(9 - 3m) = 11$
Simplifying, we have,

\[
\frac{m}{2} - 27 + 9m = 11
\]

\[
9.5m = 38
\]

\[
m = 4
\]

Substitute \( m = 4 \) into (2):

\[
3(4) + n = 9
\]

Therefore, \( n = -3 \).

Exploring another vector form — Scalar Form

Consider \( \vec{z} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \end{bmatrix} \).

Put into parametric form:

\[
z_1 = -2 - t
\]

\[
z_2 = 1 + t
\]

In each case, solve for \( t \).

\[
t = -2 - z_1 \quad (1)
\]

\[
t = -1 + z_2 \quad (2)
\]

Since these are from the same vector equation, \( t = t \):

\[
(1) = (2)
\]

\[
-2 - z_1 = -1 + z_2
\]

Solve for \( z_2 \):

\[
z_2 = -z_1 - 1
\]