1) **Exercise 1**  
In the diagram, \( \angle ABC = \angle AED \), \( AD = 3 \), \( DB = 2 \) and \( AE = 2 \). Determine the length of \( EC \).

![Diagram of triangle ABC with points A, B, C, D, and E labeled.]

**Solution:**

First, we show that \( \triangle AED \) and \( \triangle ABC \) are similar. Since \( \angle DAE = \angle BAC \) and \( \angle ABC = \angle AED \), we have that \( \triangle AED \) is similar to \( \triangle ABC \). Thus, \( \frac{DA}{AE} = \frac{CA}{AB} \). By cross multiplying,

\[
CA = \frac{DA \cdot AB}{AE} = \frac{3}{2} \cdot (3 + 2) = \frac{15}{2}.
\]

Hence, \( EC = CA - AE = \frac{15}{2} - 2 = \frac{11}{2} \).
2) In triangle $ABC$, $D$ is a point on $BC$. Further, $AB = 35$, $BD = 11$ and $AD = AC = 31$. Determine the length of $DC$.

**Solution:** Construct the Perpendicular as shown

![Diagram showing triangle ABC with point D on BC, and perpendicular from A to BC]

Now, by the Pythagorean Theorem on $\triangle ABE$, we see that

$$ (AB)^2 = (BE)^2 + (AE)^2 $$

$$ 35^2 = (11 + DE)^2 + (AE)^2 $$

$$ 35^2 = 11^2 + 22 \cdot DE + (DE)^2 + (AE)^2 $$

Now, by the Pythagorean Theorem on $\triangle ADE$, we see that

$$ (AD)^2 = (DE)^2 + (AE)^2 $$

$$ 31^2 = (DE)^2 + (AE)^2 $$

Subtracting these two equations gives

$$ 35^2 - 31^2 = 11^2 + 22 \cdot DE $$

Either using a calculator or factoring as follows, we see that

$$ 35^2 - 31^2 = 11^2 + 22 \cdot DE $$

$$ (35 - 31)(35 + 31) = 11(11 + 2 \cdot DE) $$

$$ (4)(66) = 11(11 + 2 \cdot DE) $$

$$ (4)(6) = 11 + 2 \cdot DE $$

$$ 24 - 11 = 2 \cdot DE $$

$$ 13 = DC $$

Since $2DE = DC$
3) In triangle $ABC$, $AC = AB = 25$ and $BC = 40$. Point $D$ is a point chosen on $BC$. From $D$, perpendicualrs are drawn to meet $AC$ at $E$ and $AB$ at $F$. Determine the value of $DE + DF$.

Solution: As shown in the diagram, draw the perpendicular bisector of $CB$ which intersects $A$ and $M$, the midpoint of $BC$. Also connect $A$ and $D$.

Thus, $MC = 20$ and hence by the Pythagorean Theorem,

$$(AC)^2 = (CM)^2 + (AM)^2$$

$$25^2 - 20^2 = (AM)^2$$

$$(25 - 20)(25 + 20) = (AM)^2$$

$$5 \cdot 45 = (AM)^2$$

$$5^2 \cdot 9^2 = (AM)^2$$

$$15 = AM$$

Hence, the area of $\triangle ABC$ is $\frac{AM \cdot BC}{2} = 300$. Now, denoting area by absolute values, we see that

$$300 = |\triangle ABC| = |\triangle ADC| + |\triangle ADB|$$

$$300 = \frac{AC \cdot ED}{2} + \frac{AB \cdot FD}{2}$$

$$300 = \frac{25 \cdot ED}{2} + \frac{25 \cdot FD}{2}$$

$$600 = 25(ED + FD)$$

$$24 = ED + FD$$
4) Triangle $ABC$ is an isosceles triangle in which $AB = AC = 10$ and $BC = 12$. The points $S$ and $R$ are on $BC$ such that $BS : SR : RB = 1 : 2 : 1$. The midpoints of $AB$ and $AC$ are $P$ and $Q$ respectively. Perpendiculars are drawn from $P$ and $R$ to meet $SQ$ at $M$ and $N$ respectively. What is the length of $MN$?

Solution: From the problem statement, $P$ and $Q$ are midpoints hence $AP = PB = AQ = AC = 5$. Let $X$ be the midpoint of $BC$. Since the triangle $ABC$ is isosceles, we see that $AM$ is the perpendicular bisector of $BC$. Similarly, $AY$ is the perpendicular bisector for $\triangle APQ$. Since the sides are in a $1:2:1$ ratio, we see that $BS = SX = XR = RC = 3$.

First, we claim that $QR \perp BC$. Draw the triangle below as follows.

Then $\triangle AQY$ is similar to $\triangle ACM$ (They share $\angle CAX = \angle QAY$ and both have a right angle). Hence

$$\frac{AQ}{AY} = \frac{AC}{XC} \quad \text{implying} \quad \frac{5}{AY} = \frac{10}{8}$$

Hence $AY = 4$. Similarly, $AQ = 3$. By the Pythagorean Theorem in $\triangle AMC$, we see that

$$(AC)^2 = (AX)^2 + (MC)^2$$

$$(10)^2 = (AX)^2 + 6^2$$

$$100 - 36 = (AX)^2$$

$$8 = AX$$
Hence, $XY = 4$ and by the Pythagorean Theorem again, we see that $QX = 5$. Hence, $\triangle QMC$ is isosceles and thus, $QR$ must be perpendicular since it is the bisector of $CX$.

Now, we can argue as above to show that $PS$ is perpendicular to $BC$. By the Pythagorean Theorem on $\triangle QRC$, we see that

\[(QC)^2 = (QR)^2 + (RC)^2\]
\[(5)^2 = (QR)^2 + 3^2\]
\[25 - 9 = (QR)^2\]
\[4 = QR\]

and similarly, $PS = 4$. Now, as $PS$ and $QR$ are parallel, we see that $\angle PSM = \angle SQR$. Hence $\triangle PSM$ is congruent to $\triangle RQN$ (angles are equal and they share a side length size). Thus $SM = NQ$. Since $\angle QRS$ is a right angle, we see that $\angle SRN = \angle NQR$. Further, $\angle RSN = \angle QRN$. Hence $\triangle RNQ$ which is similar to $\triangle SRQ$. This gives

\[
\frac{QN}{QS} = \frac{QR}{QS}
\]

Thus, $QN \cdot QS = 16$. Now, by Pythagorean Theorem again on $\triangle QRS$, we see that

\[(QS)^2 = (QR)^2 + (SR)^2\]
\[(QS)^2 = 4^2 + 6^2\]
\[(QS)^2 = 16 + 36\]
\[QS = \sqrt{52}\]
\[QS = 2\sqrt{13}\]

Thus, $QN = \frac{16}{2\sqrt{13}} = \frac{8}{\sqrt{13}}$. Now, $QS = SM + MN + NQ = 2QN + MN$ and so

\[MN = QS - 2QN = 2\sqrt{13} - 2 \cdot \frac{8}{\sqrt{13}} = \frac{26 - 16}{\sqrt{13}} = \frac{10}{\sqrt{13}}\]

completing the problem.
5) The lengths of the diagonals $AD$ and $BC$ in rhombus $ABCD$ are 6 and 8 respectively. Triangle $AXY$ is equilateral and line $XY$ is parallel to diagonal $BC$. Determine the length of the altitude of triangle $AXY$.

Solution: Construct the picture as shown:

Our goal is to find the length of $AF$. Lines $AC$ and $BD$ are the diagonals. Since the diagonals of a rhombus bisect each other, we see that $EC = AE = 6/2 = 3$ and $DE = BE = 8/2 = 4$. Note also that diagonals of a rhombus meet at right angles. Now, since $DB$ and $XY$ are parallel, $\angle AFY = \angle AEB = 90^\circ$. Thus, since $\angle FYA = 60^\circ$, we see that $\angle YAF = 30^\circ$. Thus, $\triangle YAF$ is a magic triangle and so

$$\frac{AF}{FY} = \frac{\sqrt{3}}{1}$$

Giving $AF = \sqrt{3} \cdot FY$ and so $AF = 3 + EF = \sqrt{3} \cdot FY$. Next, we note that $\triangle CEB$ and $\triangle CFY$ are similar since they share $\angle YCF = \angle BCE$ and $\angle CEB = \angle CFY$. Thus, by similar triangles, we see that

$$\frac{FC}{FY} = \frac{CE}{BE} = \frac{3}{4}.$$

Hence $4FC = 3FY$. Now, $FC = 3 - EF$ and thus, $12 - 4EF = 3FY$. Solving for $FY$ by substituting $EF = \sqrt{3} \cdot FY - 3$ gives

$$12 - 4(\sqrt{3} \cdot FY - 3) = 3FY$$
$$12 - 4\sqrt{3} \cdot FY + 12 = 3FY$$
$$24 = 3FY + 4\sqrt{3}FY$$
$$24 = (3 + 4\sqrt{3})FY$$
$$\frac{24}{3 + 4\sqrt{3}} = FY$$

Thus, $AF = \sqrt{3} \cdot FY = \frac{24\sqrt{3}}{3 + 4\sqrt{3}}$ completing the question.
6) In the diagram, $ABCD$ is a rhombus with $K$ the midpoint of $DC$ and $L$ the midpoint of $BC$. Segments $DL$ and $BK$ intersect at $M$. Determine the fraction of the area of quadrilateral $KMLC$ is of the area of the rhombus $ABCD$.

Solution: Denote areas by absolute values. Join the diagonal $DB$ and points $MC$ as is done in the following diagram

Now, Since $K$ is the midpoint of $DC$, we see that $|\Delta MDK| = |\Delta MKC|$. Similarly, $|\Delta MLC| = |\Delta MLB|$. Looking at $\Delta DBC$, we see that $\Delta BDK = |\Delta BKC|$. Hence, we have that

$$|\Delta BDM| + |\Delta MDK| = |KMLC| + |\Delta BML|$$

Similarly with $\Delta DBL$ and $\Delta DLC$, we see that

$$|\Delta BDM| + |\Delta BML| = |KMLC| + |\Delta MDK|$$

Subtracting these two gives

$$|\Delta MDK| - |\Delta BML| = |\Delta BML| - |\Delta MDK|$$

Which gives that $2|\Delta BML| = 2|\Delta MDK|$ and hence $|\Delta BML| = |\Delta MDK|$. This gives us that

$$|KMLC| = |\Delta MKC| + |\Delta MLC| = |\Delta MDK| + |\Delta BML| = 2|\Delta MDK|$$

Thus, in the above, we see that $\Delta BDM = |KMLC| = 2|\Delta MDK|$. Since $\Delta DBC$ is half the rhombus, we see that

$$|ABCD|/2 = |\Delta DBC| = |\Delta BDM| + |KMLC| + |\Delta MDK| + |\Delta BLM|$$

$$= 6|\Delta MDK|$$

$$= 3|KMLC|$$

and hence $KMLC$ is one sixth of the area of the rhombus.
7) Exercise 3:

In triangle $ABC$, point $D$ is on $AB$ such that $AD$ is twice as long as $DB$ and $E$ is a point on $BC$ such that $BE$ is twice as long as $BC$. If the area of triangle $ABC$ is 90 units squared, what is the area of triangle $ADE$ in units squared?

Solution: From the problem statement, we see that $BE = 2BC$. Note that $\triangle ABE$ and $\triangle AEC$ have the same heights and so their areas are in proportion with their bases. Thus, denoting area by absolute values, $|\triangle ABE| = 2|\triangle AEC|$. The problem also tells us that (suppressing units throughout)

$$90 = |\triangle ABC| = |\triangle ABE| + |\triangle AEC| = 3|\triangle AEC|$$

and hence $|\triangle AEC| = 30$. Also from the problem statement, we see that $AD = 2DB$. Note that $\triangle AED$ and $\triangle DEB$ have the same heights and so their areas are in proportion with their bases. Thus, denoting area by absolute values, $|\triangle AED| = 2|\triangle DEB|$. Since

$$60 = 2|\triangle AEC| = |\triangle ABE| = |\triangle AED| + |\triangle DEB| = 3|\triangle DEB|$$

we see that $|\triangle DEB| = 20$ and hence $|\triangle AED| = 2|\triangle DEB| = 40$. 


8) **Exercise 2**
Square $ABCD$ has an area of 4. The point $E$ is the midpoint of $AB$. Similarly, $F, G, H$ and $I$ are the midpoints of $DE, CF, DG$ and $CH$ respectively. What is the area of triangle $IDC$?

![Diagram](image)

**Solution:** As usual, denote area by absolute values. Connect $EC$ to form an isosceles triangle $DEC$. Now, since $F$ is the midpoint of $DE$, we see that $|\triangle DFC| = 0.5|\triangle DEC|$ since the median cuts the area in half. Similarly, $|\triangle DGC| = 0.5|\triangle DFC|$, $|\triangle DHC| = 0.5|\triangle DGC|$, $|\triangle DIC| = 0.5|\triangle DHC|$. Combining this gives $|\triangle DIC| = (0.5)^4|\triangle DEC| = \frac{1}{16}\cdot |\triangle DEC|$. Since the height of $\triangle DEC$ is 2 and the base is 2, it’s area is 2 and hence $|\triangle DIC| = \frac{2}{16} = \frac{1}{8}$. 
9) In the diagram, $AB = AC = 12\text{cm}$ and $AE = AD = 8\text{cm}$. The area of quadrilateral $AEFD$ is $8\text{cm}^2$. What is the area of triangle $ABC$ in square centimetres?

Solution: We suppress units until the end of this argument. Construct segment $AF$ as shown.

Now, $\triangle AEF$ and $\triangle EFB$ have the same height and their bases are in a $2:1$ proportion with each other. Further, $AF$ cuts the area of quadrilateral $AEFD$ in half by symmetry so $\triangle AEF$ has area 4. Hence, the area of $\triangle EFB$ is half the area of $\triangle AEF$ which gives the area of $\triangle EFB$ to be 2. By symmetry, $\triangle DFC$ also has area 2. Lastly, using $\triangle AEC$ which has area $8 + 2 = 10$ (adding the area of quadrilateral to the triangle $DFC$) and using $\triangle ECB$ which has area $2 + |\triangle FCB|$ (here the absolute value signs denote area), we see that since the heights of these triangles are the same and the side lengths are in a $2:1$ ration with each other, we have that $2 + |\triangle FCB| = \frac{10}{2}$ and thus $|\triangle FCB| = 3$. Adding the three triangle areas and the quadrilateral area gives $8 + 2 + 2 + 3 = 15$ square centimetres as the area.
10) The diagram shows a square $ABCD$ with unit length. Triangle $ADE$ is equilateral. The diagonal $AC$ of square $ABCD$ intersects line segment $DE$ at the point $F$. What is the area of triangle $AFD$?

Solution: Begin by constructing the following line segments on the diagram

Our goal is to find $|\triangle FDC|$, the area of $\triangle FDC$. This is given by $|\triangle FDC| = \frac{1}{2}FH \cdot DC = \frac{FH}{2}$. Now, $FGDH$ is a rectangle and hence $GD = FH$. Since $AC$ is a diagonal of a square, $\angle HCF = 45^\circ$. Since $\triangle FHC$ is a magic triangle, we see that $GD = FH = HC$. Thus, $DH = DC - HC = 1 - GD$. Since $FGDH$ is a rectangle, once again we have that $FG = DH = 1 - GD$. Since $\triangle FGD$ is a $30^\circ : 60^\circ : 90^\circ$ triangle, we see that

$$\frac{GD}{FG} = \frac{1}{\sqrt{3}}$$

$$\frac{GD}{1 - GD} = \frac{1}{\sqrt{3}}$$

$$\sqrt{3} \cdot GD = 1 - GD$$

$$(\sqrt{3} + 1)GD = 1$$

Solving gives $GD = \frac{1}{\sqrt{3} + 1}$ or $GD = \frac{\sqrt{3} - 1}{2}$. Since $GD = FH$, we see that $|\triangle FDC| = \frac{FH}{2} = \frac{\sqrt{3} - 1}{4}$ completing the question.
11) Below, $F$ is the midpoint of $AE$, $AE = \frac{1}{2}ED$, $AB = 9$ and $BC = 3$. If the area of quadrilateral $BEDC$ is 72, then what is the area of triangle $BFE$?

**Solution:** Join $BD$. Denoting area by absolute values, we see that $|\triangle BFE| = |\triangle ABF|$ since the triangles have the same base and height. Since $DE = 2AE$, we see that

$$|\triangle BED| = 2|\triangle BAE| = 2(|\triangle BFE| + |\triangle ABF|) = 2(|\triangle BFE| + |\triangle BFE|) = 4|\triangle BFE|$$

Now, since $AB : BC$ is $3 : 1$, we see that $|\triangle ABD| = 3|\triangle BCD|$. Since

$$|\triangle ABD| = |\triangle BFE| + |\triangle ABF| + |\triangle BED| = |\triangle BFE| + |\triangle BFE| + 4|\triangle BFE| = 6|\triangle BFE|$$

Combining gives $|\triangle BCD| = 2|\triangle BFE|$. Thus,

$$72 = |BEDC| = |\triangle BED| + |\triangle BCD| = 4|\triangle BFE| + 2|\triangle BFE| = 6|\triangle BFE|$$

Hence, $|\triangle BFE| = 12$. 
12) In the diagram, triangle $ABC$ is equilateral with sides of length 2. Line segments $CD$ and $EB$ are medians and $FGHI$ is a square. Determine the ratio of the area of square $FGHI$ to triangle $ABC$.

Solution: Draw the altitude $AM$ which is also the perpendicular bisector of $BC$. Let $MG = FG = n$ so that the square has side length $2n$.

From the problem statement, we know that $MC = BC/2 = 1$. Hence $GC = 1 - n$. Now, triangle $HGC$ is a magic triangle since the angles are $30^\circ : 60^\circ : 90^\circ$. Hence, we see that

\[
\frac{2n}{1-n} = \frac{HG}{GC} = \frac{1}{\sqrt{3}}
\]

This gives $2\sqrt{3}n = 1 - n$ and so $n = \frac{1}{2\sqrt{3}+1}$. Hence, the area of the square $FGHI$ is

\[
4n^2 = \frac{4}{13+4\sqrt{3}}.
\]

The area of the triangle $ABC$ is $\frac{1}{2} \cdot AM \cdot BC = \frac{1}{2} \cdot \sqrt{3} \cdot 2 = \sqrt{3}$ (note that $\triangle ABM$ is another magic triangle) and thus, the ratio of the area of this square to triangle $ABC$ is

\[
\frac{4}{\sqrt{3}} \cdot \frac{1}{12 + 13\sqrt{3}}.
\]