Math Circles - Group Theory

Question Sheet 3

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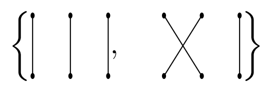
1. Consider the group Sym(3) and let H be the subgroup .
   (a) Find all the cosets of H in Sym(3).
   (b) Do the cosets of H form a group?
   (c) What is a condition that needs to be on a subgroup of a group so its cosets form a group?
2. Find all subgroups of $(\mathcal{Q}_8, \cdot)$ and compute their orders. Do the same for $(\text{Sym}(3), *)$. Any conjectures?
3. On the previous sheets, we have already seen that Braid(3) and Sym(3) have a very similar relationship to each other as $(\mathbb{Z}, +)$ and $(\mathbb{Z}_9, +)$. Recall that $(\mathbb{Z}_9, +)$ can be viewed as a group whose elements are the cosets of the subgroup $9\mathbb{Z}$ in $\mathbb{Z}$. What is the subgroup H in Braid(3) that allows us to view Sym(3) as the cosets of H?
4. Let G be a group and $H < G$ a subgroup. Prove that two elements $a, b \in G$ live in the same coset of H if and only if $a^{-1}b \in H$.
5. Let G be a finite group and $H < G$ a subgroup.
   (a) Prove that any two cosets of H have the same size.
   (b) Prove that every element of G belongs to a coset of H.
   (c) Prove that for any two cosets of H, they are either disjoint, or one is contained entirely in the other.

   Even if you didn't prove (a),(b), and (c), what can you deduce from these facts?
6. A group G has a **generator** if there is an element a such that $|a| = |G|$. Which of the following groups has a generator? $(\mathbb{Z}_4, +)$, $(\mathbb{Z}_8^*, \times)$, $(\{1, -1, i, -i\}, \times)$. Can you say anything about which of these groups are isomorphic? Which aren't? Why would such an element be called a generator?
7. (a) Prove that the order of an element must divide the order of the group.
   (b) Prove Fermat's little theorem: If p is a prime number, then $a^p \equiv a \mod p$.
8. (a) What are the possible orders of the subgroups of $(\mathbb{Z}_{12}, +)$? For each possible order, does there exist a subgroup with that order? What is it?
   (b) Answer part (a) for $(\mathbb{Z}_n, +)$.
9. Go back and answer any questions from the first two sheets.
10. Look carefully at everything we've done so far and compare it to your list of conjectures. Were your guesses correct? If you still have unanswered questions, try to figure them out.