Recurrences, Part 3

Troy Vasiga

Centre for Education in Mathematics and Computing
Faculty of Mathematics, University of Waterloo

cemc.uwaterloo.ca

Outline

- Selected solutions to problems from last week
- L-systems
- Examples of L-systems
- Prouhet-Thue-Morse sequence
- Other definitions of PTM
- Morphisms
- Squares, cubes and overlaps
- Summary

Solution to Problem 5

Use the Master Theorem to find the asymptotic running time for the recurrence

$$T(n) = 4T(n/2) + n.$$

Solution:
We have $a = 4$, $b = 2$ and $f(n) = n$.

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We calculate $n^{\log_b a} = n^{\log_2 4} = n^2$.
Then, $f(n) = n \in O(n^{2-\epsilon}$ for $\epsilon = 0.9$ (for instance).
Thus, we are in case 1 of the Master Theorem, and thus
$T(n) = \Theta(n^{\log_b a}) = \Theta(n^2)$.

Solution to Problem 7

Use the Master Theorem to find the asymptotic running time for the recurrence

$$T(n) = 4T(n/2) + n^3.$$

Solution:

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We calculate $n^{\log_b a} = n^{\log_2 4} = n^2$.

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Then, $f(n) = n^3 \in \Omega(n^{2+\epsilon})$ for $\epsilon = 1$, for instance.

Solution to Problem 7

Use the Master Theorem to find the asymptotic running time for the recurrence

$$T(n) = 4T(n/2) + \boxed{n^3}.$$

Solution:

We have $a = 4$, $b = 2$ and $f(n) = n^3$.

We calculate $n^{\log_b a} = n^{\log_2 4} = n^2$.

Then, $f(n) = n^3 \in \Omega(n^{3-\epsilon})$ for $\epsilon = 1$, for instance.

Thus, we are in case 3 of the Master Theorem. We just need to verify that the second condition holds: that is, we need to show that

$$af\left(\frac{n}{b}\right) \leq cf(n)$$

for some fixed c for all n sufficiently large.

Solution to Problem 7 (continued)

Plugging in our constants and $f(n) = n^3$, we have to show:

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Pick $c = 1$ and then for $n > 1$, we have:

$$\frac{n^3}{2} \leq cn^3.$$

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Plugging in our constants and $f(n) = n^3$, we have to show:

$$4\left(\frac{n}{2}\right)^3 \leq cn^3.$$

Pick $c = 1$ and then for $n > 1$, we have:

$$\frac{n^3}{2} \leq cn^3.$$

Thus, by case 3 of the Master Theorem, $T(n) = \Theta(n^3)$.

Definition of an L-system

An *L-system* or *Lindenmayer system* is a parallel rewriting system.

By parallel, we mean we each step involves replacing every possible occurrence at the same time.

We also need to specify the starting condition/symbol.

Rewriting Rules

We use the rule

$$X \to Y$$

to mean "replace every occurrence of X with Y."

For example, if we have the rule $X \to ABX$ and we have the current word:

$$\boxed{ABXBAX}$$

then the next word will be

Rewriting Rules

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$$ABXBAX$$

then the next word will be

$$ABABXBAABX$$

What does this sequence do?

1. $A \to AB$
2. $B \to \epsilon$

(Here ϵ means "this disappears.")

Start with A $\Rightarrow AB \Rightarrow AB \Rightarrow AB$

Not much

This just generates AB at the first step, which becomes AB after the second step, and so on.

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$f(g(x))$
$\equiv f \circ g(x)$

Notice: if $f(t) = t$, then $f(f(t)) = f(t) = t$, which means $f^n(t) = f(f(\cdots f(t))\cdots) = t$.

What does this sequence do?

1. $A \to AB$
2. $B \to B$

Start with A $\overset{1}{\Rightarrow}$ $AB \Rightarrow ABB \Rightarrow ABBB \Rightarrow \ldots$

Infinitely more

Infinitely more

After n iterations of the rewriting rules, we have the word

$$AB^n = A\underbrace{BB\cdots B}_{n}.$$

What does this sequence do?

1. $A \to B$
2. $B \to AB$

Start with A.

A	B		
1	0	1	A
0	1	1	B
1	1	2	AB
1	2	3	BAB
2	3	5	ABBAB
3	5	8	BABABBAB

Hint: write out the first few terms and look at them.

Fibonacci

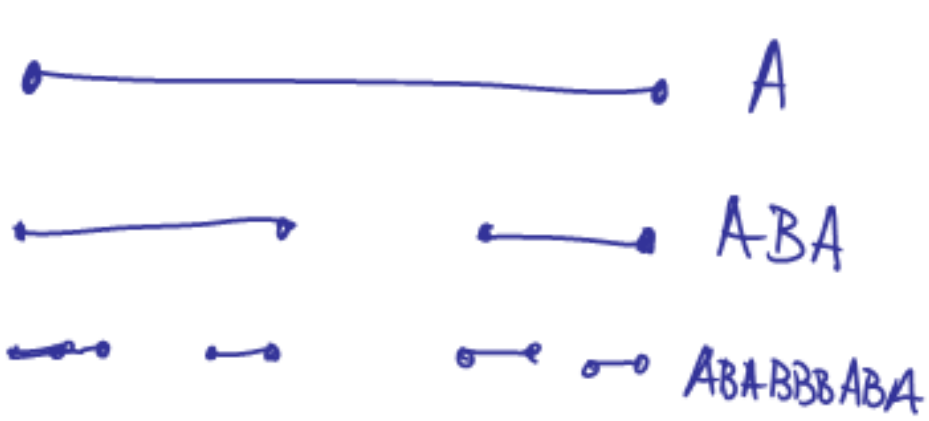
Notice the length.

Notice the ratio

$$\frac{|A|}{|B|} = \frac{f_{n-1}}{f_{n-2}} \leftarrow \lim_{n \to \infty} = \varphi$$

What does this sequence do?

1. $A \to ABA$
2. $B \to BBB$

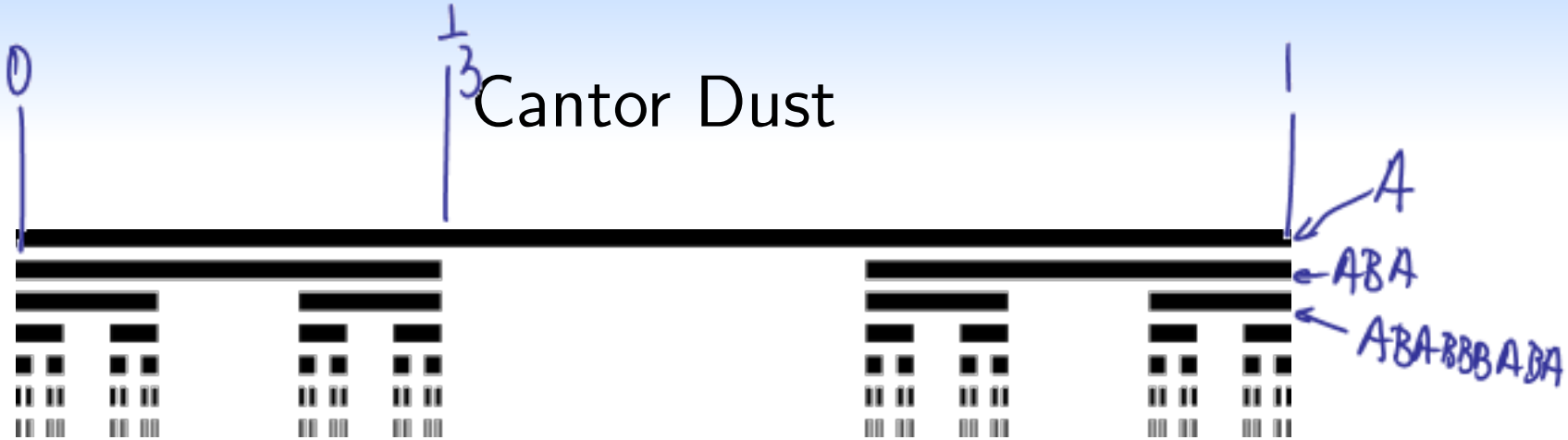


Start with A. $\Rightarrow$ ABA $\Rightarrow$ ABA BBB ABA

$\Rightarrow$ ABA BBB ABA BBBBBBBBBABABBBABA

Hint: draw a straight line.

Cantor Dust



How much of the line is removed?

A = line

B = no line

$$\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \dots = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n}$$

$$\frac{a}{1-r} = \frac{\frac{1}{3}}{1-\frac{2}{3}} = \frac{\frac{1}{3}}{\frac{1}{3}} = 1 = \frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n-1}$$

What about the number $\frac{1}{4}$ for instance?

never removed $\frac{2}{9} < \frac{1}{4} < \frac{1}{3}$, ...

What does this sequence do?

1. $X \to X + YF$
2. $Y \to FX - Y$

Start with FX. $\Rightarrow FX + YF$

$\Rightarrow FX + YF + FX - YF$

$\Rightarrow FX + YF + FX - YF + FX + YF - FX - YF$

$F + F + F - F + F + F - F - F$

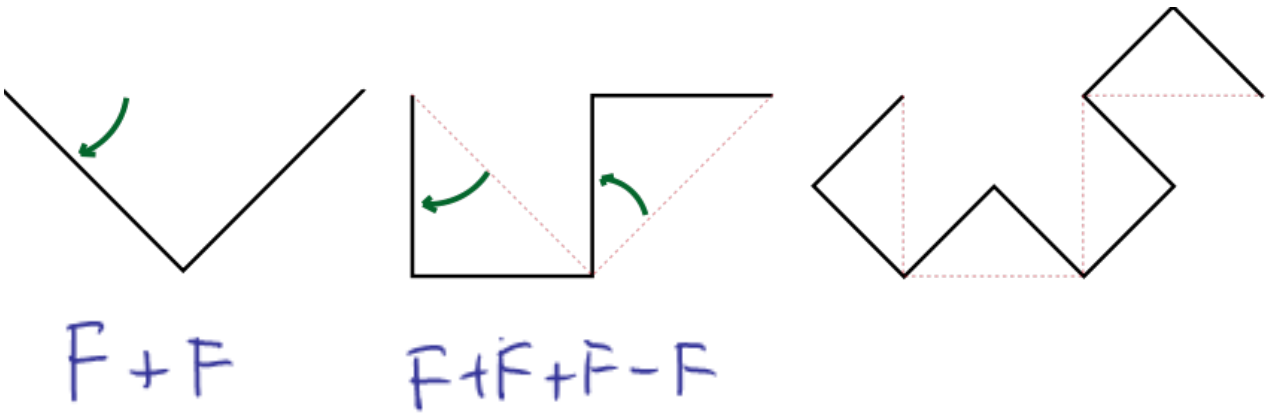
Dragon Curve

Do the following things:

- F means move forward
- $+$ means turn clockwise 90°
- $-$ means turn counterclockwise 90°

Dragon Curve

Here are some steps:



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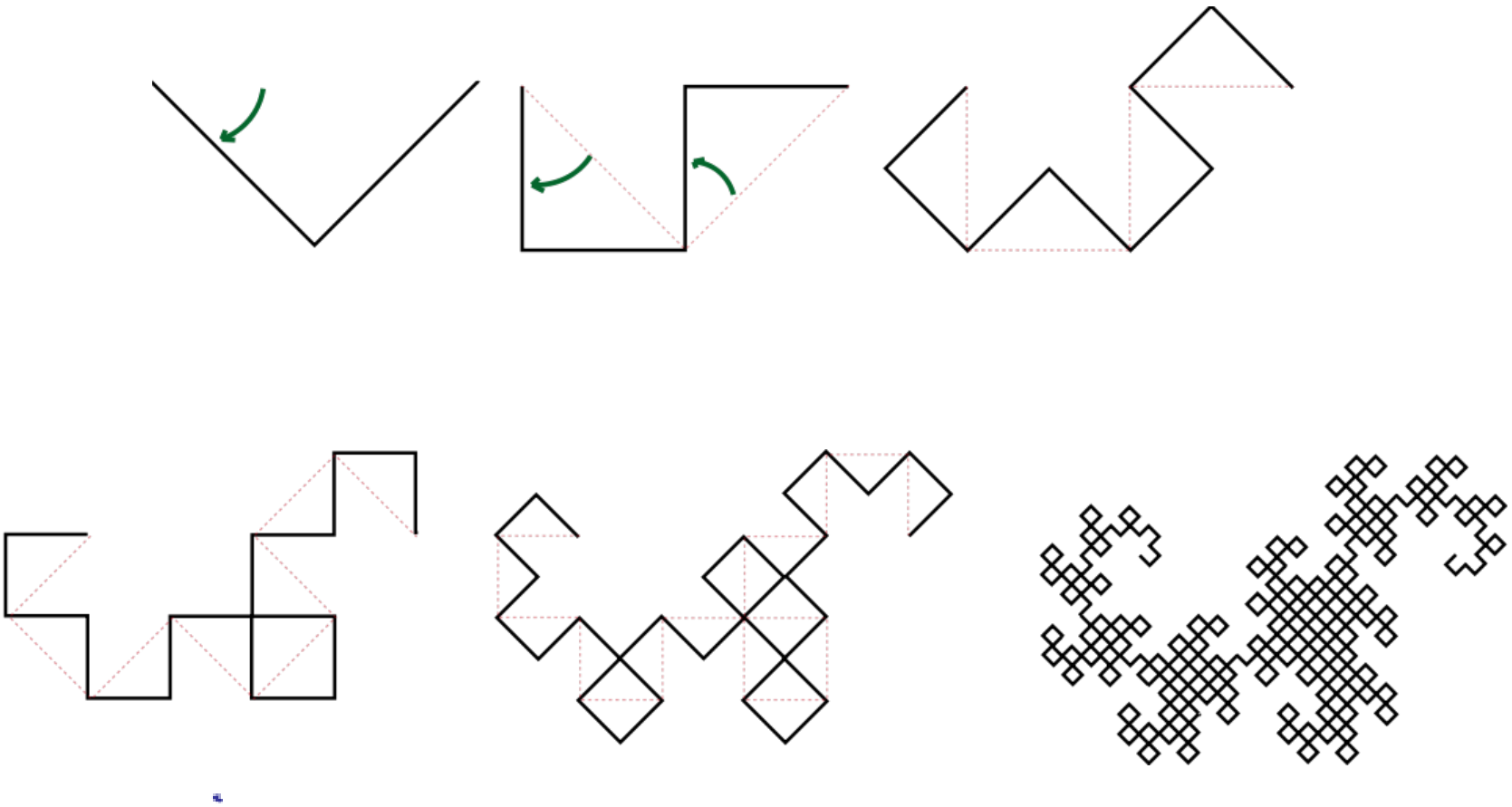
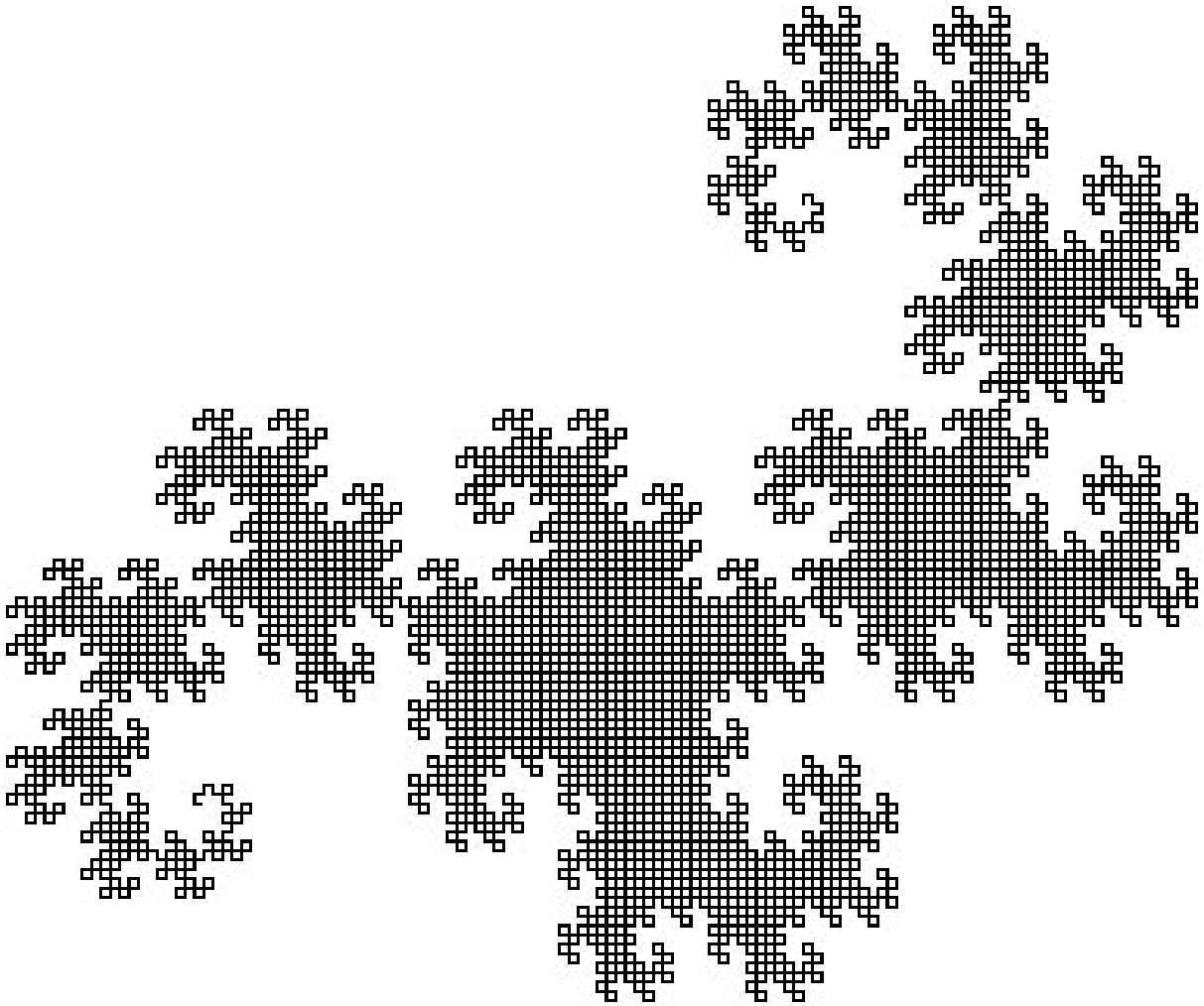


Dragon Curve



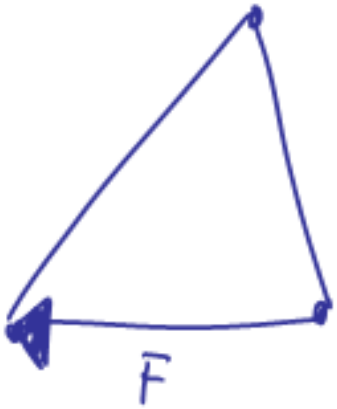
Another L-system

Start with:

$F - -F - -F$ 0

One rule:

$F \to F + F - -F + F$



F+F - - F+F
- -
F+F - - F+F
- -
F+F - - F+F

1

Another L-system

Start with:

$$F - -F - -F$$

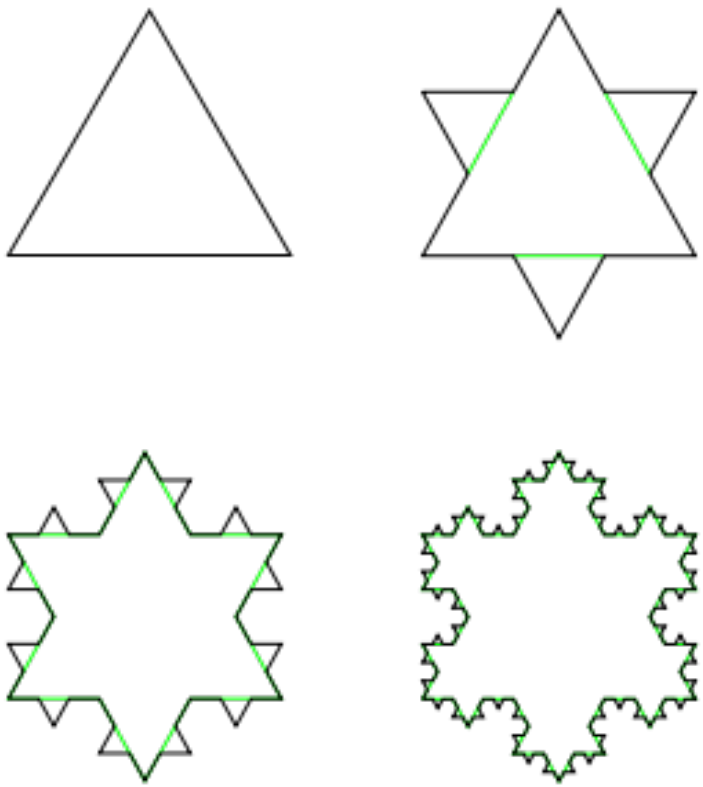
One rule:

$$F \to F + F - -F + F$$

Let:

- F: move forward
- $+$: move clockwise by $\frac{\pi}{3}$
- $-$: move counterclockwise by $\frac{\pi}{3}$

Koch Snowflake



What does this sequence do?

1. $0 \to 01$
2. $1 \to 10$

Start 0. $\Rightarrow 01 \Rightarrow \underline{0110} \Rightarrow \underline{01101001}$

$\Rightarrow \underline{0110\,1001}\,1001\,0110$

Prouhet-Thue-Morse Sequence

It does lots of things.

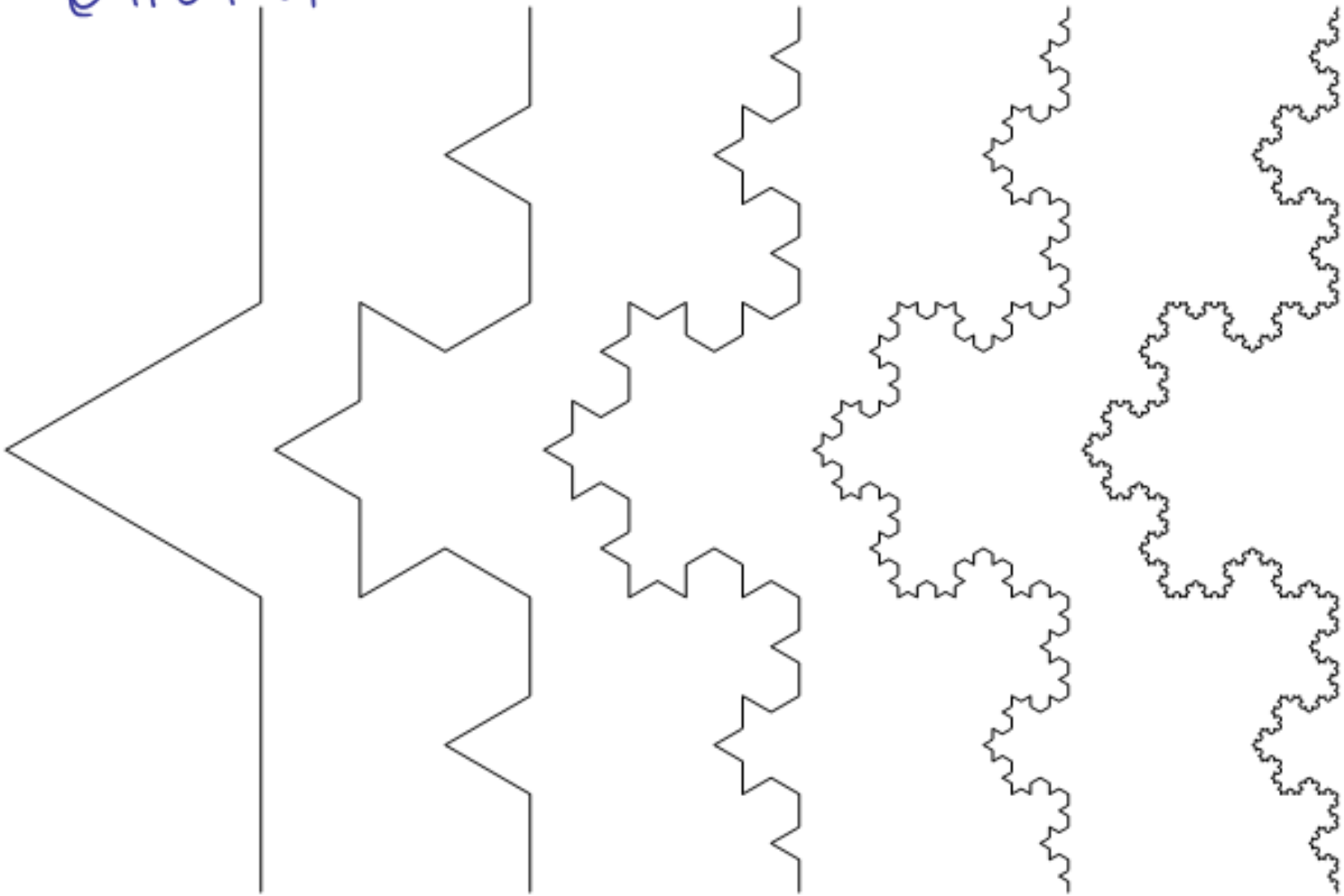
Call the sequence $\mathbf{t} = t_0 t_1 t_2 ... = 0\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ ...$

Draw a picture

- If $t_n = 0$, turn by π
- If $t_n = 1$, move ahead one unit and then rotate counterclockwise by an angle of $\frac{\pi}{3}$.

Koch Curve

0110100 1



Another definition of the PTM Sequence

Write out all integers in base-2 (i.e., binary).

0, 1, 10, 11, 100, 101, 110, 111, 1000, 1001

Define $s_2(x)$ to the sum of the digits in the base-2 representation of x.

0 1 1 0 1 2 2 3 1 2

0 0 1 0

[0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0]

Yet another definition of the PTM Sequence

Write the number, then flip the bits and rewrite it.

Define:

$$\boxed{\begin{aligned} X_0 &= 0 \\ X_{n+1} &= X_n \bar{X}_n \end{aligned}}$$

where $\bar{x}$ means change all 0's to 1's, and 1's to 0's.

0
01
0110

0110 1001

0110 1001 1001 0110

Equivalent definitions

Prove that the last two definitions are equivalent. In other words, prove:

$$t_n = s_2(n) \mod 2$$

*See PDF

Proof: Use induction.

Base case: $n = 0$. $t_0 = 0$ $s_2(0) = 0$

Assume true for all $n < n'$.

Let k be the integer such that $\boxed{2^k} \leq n < \boxed{2^{k+1}}$.

we know # of bits

Morphism

A *morphism* is a map h on strings that satisfies the identity

$$h(xy) = h(x)h(y)$$

for all strings x, y.

PTM morphism

Define the PTM morphism

$$\mu(0) = 01,$$

$$\mu(1) = 10.$$

Then

$$\mu(0) \quad = \quad 01$$

PTM morphism

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$$\mu(0) = 01,$$

$$\mu(1) = 10.$$

Then

$$\begin{aligned} \mu(0) &= 01 \\ \mu^2(0) = \mu(\mu(0)) &= 0110 \end{aligned}$$

PTM morphism

Define the PTM morphism

$$\left.\begin{aligned}\mu(0) &= 01,\\ \mu(1) &= 10.\end{aligned}\right] \Longleftrightarrow$$

L-system

$0 \to 01$

$1 \to 10$

Then

$$\begin{aligned}\mu(0) &= 01\\ \mu^2(0) = \mu(\mu(0)) &= 0110\\ \mu^3(0) &= 01101001\\ \mu^4(0) &= 0110100110010110\end{aligned}$$

Fixed point

little omega

The *fixed point of a morphism h beginning with a* is:

$$h^{\omega}(a) = \lim_{m \to \infty} h^{m}(a).$$

Equivalent definition

Let's prove $\mu^n(0) = X_n$.

Use mathematical induction.

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Prove true for $n = k + 1$.

$$\mu^{k+1}(0) \quad =$$

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$$\mu^{k+1}(0) \quad = \quad \mu^k(\mu(0))$$

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$$\begin{aligned} \mu^{k+1}(0) &= \mu^k(\mu(0)) \\ &= \end{aligned}$$

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$$\begin{aligned} \mu^{k+1}(0) &= \mu^k(\mu(0)) \\ &= \mu^k(01) \\ &= \mu^k(0)\mu^k(1) \end{aligned}$$

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&= X_k\bar{X}_k \\
&= X_{k+1}
\end{aligned}$$

Overlap-free

murmur

mama

papa

dada

hotshots

A *square* is a word of the form xx.

A *cube* is a word of the form xxx. lalala

An *overlap* is a word $axaxa$, with a single letter a and x being any word.

English word examples?

mamam

Overlap-free

0110 100 1001 0110

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A *cube* is a word of the form xxx.

no 111, 000

no 01010, 10101

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English word examples?

How does this relate to the Thue-Morse sequence?

it is overlap free

What does this sequence converge to?

Consider the sequence:

$$\frac{1}{2}, \frac{1/2}{3/4}, \frac{\frac{1/2}{3/4}}{\frac{5/6}{7/8}}, \frac{\frac{\frac{1/2}{3/4}}{\frac{5/6}{7/8}}}{\frac{\frac{9/10}{11/12}}{\frac{13/14}{15/16}}}$$

What does this converge to?

Proof

First, observe the limit is:

$$\Pi_{n\geq 0}\left(\frac{2n+1}{2n+2}\right)^{(-1)^{t_n}}$$

Let:

$$P = \Pi_{n\geq 0}\left(\frac{2n+1}{2n+2}\right)^{(-1)^{t_n}}$$

$$Q = \Pi_{n\geq 0}\left(\frac{2n}{2n+1}\right)^{(-1)^{t_n}}$$

Notice that

$$PQ = \frac{1}{2}\Pi_{n\geq 1}\left(\frac{n}{n+1}\right)^{(-1)^{t_n}}$$

Proof

$$PQ = \frac{1}{2}\Pi_{n\geq 1}\left(\frac{n}{n+1}\right)^{(-1)^{t_n}}$$

Rewrite, breaking over odd and even indices:

$$\begin{aligned} PQ &= \frac{1}{2}\Pi_{n\geq 0}\left(\frac{2n+1}{2n+2}\right)^{(-1)^{t_{2n+1}}}\Pi_{n\geq 1}\left(\frac{2n}{2n+1}\right)^{(-1)^{t_n}} \\ &= \frac{1}{2}P^{-1}Q. \end{aligned}$$

Thus, $P^2 = \frac{1}{2}$.

Summary

- recursion is very natural
- proving results using recursion is powerful
- recursion has a self-contained, succinct form
- recursion is beautiful

math = recursion