1. In the diagram, $AB$ is parallel to $CD$.

Determine the values of $x$ and $y$.

**Solution**

Let $a = \angle EGF$, $b = \angle FEG$. Since $AB \parallel CD$, by $Z$ pattern, $a = 50^\circ$. Observe that $a$ and the angle $13x$ form a straight angle. Then,

\[ a + 13x = 180 \]
\[ 50 + 13x = 180 \]
\[ x = 10 \]

Similarly, using $b$, the $50^\circ$ angle, and the angle $3x = 30$,

\[ b + 50 + 30 = 180 \]
\[ b + 80 = 180 \]
\[ b = 100 \]

Since $y$ is an external angle to $\triangle EFG$, $y = a + b = 150^\circ$.

Therefore, $x = 10^\circ$, $y = 150^\circ$.

2. Triangle $ABC$ has a right angle at $B$. $AC$ is extended to $D$ so that $CD = CB$. The bisector of angle $A$ meets $BD$ at $E$. Prove that $\angle AEB = 45^\circ$.

**Solution**

Since $AE$ bisects $\angle BAC$, we can let $x = \angle BA E = \angle EAC$.

Since $CB = CD$, $\triangle BCD$ is isosceles so $y = \angle CBD = \angle CDB$.

In $\triangle ABD$, by the sum of interior angles of a triangle,

\[ 2x + (90 + y) + y = 180 \]
\[ 90 + 2x + 2y = 180 \]
\[ 2x + 2y = 90 \]
\[ x + y = 45 \]

In $\triangle ABE$, using the sum of interior angles,

\[ x + (90 + y) + \angle AEB = 180 \]
\[ 90 + (x + y) + \angle AEB = 180 \]
\[ 90 + 45 + \angle AEB = 180 \]
\[ 45 + \angle AEB = 90 \]
\[ \angle AEB = 45^\circ \] (as required)
3. In the diagram, $AB$ is parallel to $DC$ and $AB = BD = BC$. If $\angle A = 52^\circ$, determine the measure of $\angle DBC$.

**Solution**

$\triangle ABD$ is isosceles since $AB = BD$. Therefore $\angle BDA = \angle BAD = 52^\circ$.

Then in $\triangle BAD$, 

$$\angle ABD = 180^\circ - \angle A - \angle BDA = 180^\circ - 52^\circ - 52^\circ = 76^\circ$$

Since $AB \parallel DC$, we have $\angle BDC = \angle ABD = 76^\circ$.

Since $BD = BC$, $\triangle BDC$ is isosceles. Therefore, $\angle BDC = \angle BCD = 76^\circ$.

Therefore, by sum of interior angles of a triangle, $\angle DBC = 180^\circ - 76^\circ - 76^\circ = 28^\circ$.

4. The diagram shows three squares of the same size. What is the value of $x$?

**Solution**

In a square, the corner angles are $90^\circ$. The triangle is equilateral (all sides equal), so we know all the angles are equal and hence must be $60^\circ$ each.

If we look at the place where the triangle and two squares meet (where $x$ is located), we notice it is made up of four angles; two corner angles of a square, one corner angle of a triangle, and $x$. These four angles form a complete revolution, so they must sum up to $360^\circ$.

Then,

$$x + 90 + 90 + 60 = 360$$
$$x + 240 = 360$$
$$x = 120^\circ$$

Therefore the measure of angle $x$ is $120^\circ$. 
5. The diagram shows a rhombus $FGHI$ and an isosceles triangle $FGJ$ in which $GF = GJ$. Angle $FJI$ equals $111^\circ$. What is the measure of angle $JFI$?

Solution

Since $\angle FJI = 111^\circ$ is part of a straight angle with $\angle FJG$, we have that $\angle FJG = 69^\circ$.

We see that because $GF = GJ$, $\triangle FGJ$ is isosceles, with equal base angles $\angle FJG$ and $\angle GFJ$, we get $\angle GFJ = 69^\circ$ and so $\angle FJI = 42^\circ$.

Because $FG \parallel IH$, $\angle FGI = \angle GIH = 42^\circ$. Also, $\triangle IHG$ is isosceles since $GH = HI$, so $\angle IGH = \angle GIH = 42^\circ$.

Since $GH \parallel FI$, $\angle FIG = \angle IGH = 42^\circ$.

Using $\triangle FJI$, we see

\[
\angle FJI + \angle FIJ + \angle JFI = 180
\]

\[
111 + 42 + \angle JFI = 180
\]

\[
\therefore \angle JFI = 27^\circ
\]

6. $ABCD$ is a square. The point $E$ is outside the square so that $CDE$ is an equilateral triangle. Find angle $BED$.

Solution

Since $ABCD$ is a square, $BC = CD$. Since $\triangle CDE$ is equilateral, $CD = DE = EC$. Therefore, $BC = CD = DE = EC$ and so $BC = EC$.

By the properties of a square, $\angle BCD = 90^\circ$. By the properties of equilateral triangles, $\angle DCE = 60^\circ$. Therefore $\angle BCE = \angle BCD + \angle DCE = 90 + 60 = 150^\circ$.

Since $BC = EC$, $\triangle BCE$ is isosceles. So $\angle EBC = \angle BEC = x$. In this triangle, we have

\[
\angle BCE + x + x = 180
\]

\[
150 + 2x = 180
\]

\[
x = 15^\circ
\]

So $\angle BEC = x = 15^\circ$.

Note that $60^\circ = \angle DEC = \angle BED + \angle BEC = \angle BED + 15$.

Therefore, $\angle BED = 60 - 15 = 45^\circ$. 

7. The diagram shows two isosceles triangles in which the four angles marked $x$ are equal. The two angles marked $y$ are also equal. Find an equation relating $x$ and $y$.

**Solution**

Consider the angles opposite to the angles marked $y$. Since they are opposite angles, they are equal to $y$.

The quadrilateral formed in the overlap must have angle sum $360^\circ$. We know two of the angles are $y$.

The other two angles are actually the missing angle of the two isosceles triangles. In the left triangle, this angle is $180 - 2x$; for the triangle on the right, it is also $180 - 2x$.

These four angles have to sum to $360^\circ$. Therefore,

$$y + y + (180 - 2x) + (180 - 2x) = 360$$

$$2y + 360 - 4x = 360$$

$$2y = 4x$$

$$y = 2x$$

$\therefore y = 2x$ is our desired relationship.

8. In the diagram, $QSR$ is a straight line.

$\angle QPS = 12^\circ$ and $PQ = PS = RS$. What is the size of $\angle QPR$?

**Solution**

Let $\angle SPR = x$. Then, $\angle QPR = \angle QPS + \angle SPR = 12^\circ + x$.

Since $PS = SR$, $\triangle SPR$ is isosceles and so $\angle PRS = \angle SPR = x$. Since $PS = PQ$, $\triangle PQS$ is isosceles and so $\angle PQS = \angle PSQ = y$.

Then

$$12 + y + y = 180$$

$$2y = 168$$

$$y = 84^\circ$$

Since $QSR$ is a straight line, $y = \angle PSQ$ is external to $\triangle PSR$, so $84^\circ = y = x + x = 2x$. Therefore, $x = 42^\circ$. 


9. The diagram shows a regular nonagon with two sides extended to meet at point $X$. What is the size of the acute angle at $X$?

![Diagram of a regular nonagon with two sides extended to meet at point X]

**Solution**

In a regular nonagon (9 sides), the sum of the interior angles is $(9 - 2) \times 180^\circ = 1260^\circ$. Since the figure is regular, all the interior angles are equal. \( \therefore \) each angle is \( \frac{1260^\circ}{9} = 140^\circ \).

Using our diagram, the two extended sides each form a straight angle. One part of each straight angle is the interior angle, $140^\circ$. The other part we will call $x$ must be $40^\circ$.

$y$ is part of a revolution; the other part of the revolution is one interior angle of the nonagon, $140^\circ$. So $y = 220^\circ$.

The shape containing the angles $X, x, y$ is a quadrilateral. The interior sum must therefore be $360^\circ$.

So, $X + x + x + y = 360^\circ$. Plugging in our values for $x, y$, we see

\[
X = 360 - 2x - y = 360 - 80 - 220 = 60^\circ
\]

Therefore, $X = 60^\circ$.

10. The three angle bisectors of triangle $LMN$ meet at a point $O$ as shown. Angle $LNM$ is $68^\circ$. What is the size of angle $LOM$?

![Diagram of a triangle with angle bisectors]

**Solution**

Since we are using angle bisectors, let $\angle LNO = \angle ONM = x$, $\angle NLO = \angle OLM = y$, and $\angle LMO = \angle OMN = z$.

But $68^\circ = \angle LNM = \angle NLO + \angle OLM = 2x$, so $x = 34^\circ$.

We also have $\angle LON = 180 - (x + y) = 146 - y$, $\angle LOM = 180 - (y + z)$, and $\angle NOM = 180 - (x + z) = 146 - z$.

\( \angle LON, \angle NOM, \) and $\angle LOM$ form a complete revolution.

So, $\angle LOM = 360 - \angle LON - \angle NOM = 360 - (146 - y) - (146 - z) = 68 + y + z$

Using the entire triangle,

\[
\angle LNM + \angle NLM + \angle LMN = 180 \\
68 + 2y + 2z = 180 \\
2y + 2z = 112 \\
y + z = 56
\]

Therefore, substituting back in, we get $\angle LOM = 68 + 56 = 124^\circ$. 

5
11. In the figure shown, $AB = AF$ and $ABC$, $AFD$, $BFE$, and $CDE$ are all straight lines. Determine an equation relating $x$, $y$ and $z$.

**Solution**

Since $AB = AF$, $\triangle ABF$ is isosceles, so $\angle AFB = \angle ABF = a$.

Since $\angle AFB$ and $\angle DFE$ are opposite angles, $\angle DFE = \angle AFB = a$.

$\angle ABE$ is external to $\triangle CBE$, so $\angle ABE = \angle ACE + \angle BEC$ and $a = x + z$ follows. (1)

$\angle ADC$ is external to $\triangle DFE$, so $\angle ADC = \angle DFE + \angle DEC$ and $y = a + z$ follows. (2)

Substituting (1) into (2) for $a$, we obtain $y = x + z + z$. Rearranging and simplifying we obtain $x - y + 2z = 0$. This is the equation relating $x$, $y$, $z$.

12. The angles of a nonagon are nine consecutive numbers. What are these numbers?

**Solution**

In problem 9, we determined that the sum of the interior angles of a nonagon is $1260^\circ$.

Order the angles from least to greatest, and let the middle angle (the 5th) be $x$. Since they are consecutive numbers, the angles are

$$\{x - 4, x - 3, x - 2, x - 1, x, x + 1, x + 2, x + 3, x + 4\}$$

Summing these angles should give us $1260^\circ$. If you add the nine angles, you get $9x$. So $9x = 1260^\circ$. $\therefore x = 140^\circ$. This is the fifth angle.

Therefore, the list of angles is $\{136^\circ, 137^\circ, 138^\circ, 139^\circ, 140^\circ, 141^\circ, 142^\circ, 143^\circ, 144^\circ\}$.
13. What is the measure of the angle formed by the hands of a clock at 9:10?

Solution

Every minute after the hour, the minute hand moves \( \frac{360}{60} = 6^\circ \) from 12 o’clock. So after 10 minutes, it has moved \( 10 \times 6 = 60^\circ \) past 12 o’clock.

In one hour, the hour hand moves \( \frac{360}{12} = 30^\circ \). In ten minutes, it will have moved \( \frac{1}{6} \) of this, so it has moved \( \frac{1}{6} \times 30 = 5^\circ \) closer to 12 o’clock. 9 o’clock is located 90° before 12 o’clock, so the hour hand will be 85° before 12 o’clock.

Therefore, the total angle between the hour and minute hand will be 85 + 60 = 145°.

14. Determine the sum of the angles \( A, B, C, D, \) and \( E \) in the five-pointed star shown.

Solution

\[ a, b, c, d, e \] are exterior angles of a pentagon. So they sum to 360°. \( f, g, h, i, j \) are also exterior angles, so they also sum to 360°.

If we add up all the letters in the diagram, we are adding up all the interior angles of five triangles. So the total sum should equal \( 5 \times 180 = 900^\circ \).

Doing this gives,

\[
\begin{align*}
  a + b + c + d + e + f + g + h + i + j + r + s + t + u + v &= 900 \\
  (a + b + c + d + e) + (f + g + h + i + j) + r + s + t + u + v &= 900 \\
  (360) + (360) + r + s + t + u + v &= 900 \\
  r + s + t + u + v &= 180 \\
  \therefore A + B + C + D + E &= 180^\circ
\end{align*}
\]
15. In $\triangle PQR$, $PQ = PR$. $PQ$ is extended to $S$ so that $QS = QR$. Prove that $\anglePRS = 3(\angle QSR)$.

Solution

Since $PQ = PR$ and $QS = QR$, we can label the diagram as above.

Note that $\angle SPR = 180 - 2y$. Using $\triangle SPR$, we see the angle sum gives us

\[
180 = \angle SPR + \angle PSR + \angle PRS \\
180 = (180 - 2y) + x + (x + y) \\
180 = 180 - y + 2x \\
y = 2x
\]

So $\angle PRS = x + y = x + 2x = 3x = 3(\angle QSR)$ as required.

16. A regular pentagon is a five-sided figure which has all of its angles equal and all of its side lengths equal. In the diagram, $TREND$ is a regular pentagon, $PEA$ is an equilateral triangle, and $OPEN$ is a square. Determine the size of $\angle EAR$.

Solution

Since $\triangle APE$ is equilateral, $\angle PEA = 60^\circ$.

Since $OPEN$ is a square, $\angle PEN = 90^\circ$.

Since $TREND$ is a regular pentagon, with interior angle sum is $540^\circ$, each angle equals $540 \div 5 = 108^\circ$. So $\angle NER = 108^\circ$. At $E$, the angles make a complete rotation, so

\[
\angle AER = 360 - \angle PEA - \angle PEN - \angle NER \\
= 360 - 60 - 90 - 108 \\
= 102^\circ
\]

Since $\triangle APE$ is equilateral, $AE = PE$. Since $OPEN$ is a square, $PE = EN$. Since $TREND$ is a regular pentagon, $EN = ER$. Therefore $AE = PE = EN = ER$ and $\triangle EAR$ is isosceles. It follows that $\angle EAR = \angle ERA = x$.

In $\triangle EAR$, we then have

\[
\angle EAR + \angle ERA + \angle AER = 180^\circ \\
x + x + 102 = 180 \\
2x = 78 \\
x = 39
\]

Therefore, $\angle EAR = 39^\circ$
17. A beam of light shines from point $S$, reflects off a reflector at point $P$, and reaches point $T$ so that $PT$ is perpendicular to $RS$. What is the value of $x$?

**Solution**

Extend $TP$ to $RS$, intersecting $RS$ at the point $Q$ as in the diagram. Then $\triangle PQS$ is a right triangle.

Since $\angle TPS$ is exterior to $\triangle PQS$, $\angle TPS = 90 + 26 = 116^\circ$.

Since the reflector forms a straight line, the two angles marked $x$ and $\angle TPS$ form a straight angle. Then

\[
\angle TPS + x + x = 180^\circ \\
116 + 2x = 180 \\
2x = 64 \\
\therefore x = 32^\circ
\]

18. In the diagram, let $M$ be the point of intersection of the three altitudes of triangle $ABC$. If $AB = CM$, then what is $\angle BCA$ in degrees?

**Solution**

Let the three altitudes be $AD$, $BE$ and $CF$. In $\triangle CFB$ and $\triangle ADB$, we have $\angle CFB = \angle ADB = 90^\circ$.

Also, $\angle CBF$ and $\angle DBA$ are the same angle, so $\triangle CFB \sim \triangle ADB$.

$\therefore \angle DAB = \angle FCB = x$.

Applying the same argument to $\triangle CFA$ and $\triangle BEA$, we get $\angle FCA = \angle EBA = y$.

In $\triangle CDM$ and $\triangle ADB$,

\[
\angle DCM = \angle DAB = x \\
\angle CDM = \angle ADB = 90^\circ \\
\therefore \angle CMD = \angle DBA \\
CM = BA \\
\therefore \triangle CDM \cong \triangle ADB$ and $CD = DA$

So $\triangle CDA$ is right isosceles, hence $\angle DCA = \angle DAC = 45^\circ$. Therefore $\angle BCA = 45^\circ$, since $\angle DCA = \angle BCA$. 
19. In the diagram, $PW$ is parallel to $QX$, $S$ and $T$ lie on $QX$, and $U$ and $V$ are the points of intersection of $PW$ with $SR$ and $TR$, respectively. If $\angle SUV = 120^\circ$ and $\angle VTX = 112^\circ$, what is the measure of $\angle URV$?

**Solution**

Since $PW \parallel QX$, we have

$$\angle SUV + \angle TSU = 180^\circ$$

$$120 + \angle TSU = 180$$

$$\angle TSU = 60^\circ$$

$\angle RTX$ is exterior to $\triangle RST$. $\therefore \angle RTX = \angle SRT + \angle RST$. (1)

But $\angle RTX = \angle VTX = 112^\circ$ (same angle, given info)

and $\angle RST = \angle TSU = 60^\circ$ (same angle)

$\therefore$, substituting in (1), we have

$$\angle SRT + 60 = 112$$

$$\angle SRT = 52^\circ$$

But $\angle SRT$ and $\angle URV$ are the same angle. $\therefore \angle URV = 52^\circ$.

20. Three regular polygons meet at a point and do not overlap. One has 3 sides and one has 42 sides. How many sides does the third polygon have? Can you find other sets of three polygons that have this property?

**Solution**

Each angle in a regular 3 sided polygon is $\frac{180^\circ}{3} = 60^\circ$.

Each angle in a regular 42 sided polygon is $\frac{180(42 - 2)}{42} = \frac{1200^\circ}{7}$.

Each angle in a regular n-gon is $\frac{180(n - 2)}{n}$.

The 3 angles form a complete revolution.

$$\therefore 60^\circ + \frac{1200^\circ}{7} + \frac{180(n - 2)}{n} = 360^\circ$$

$$\frac{180(n - 2)}{n} = 360^\circ - 60^\circ - \frac{1200^\circ}{7}$$

$$\frac{180(n - 2)}{n} = \frac{900^\circ}{7}$$

$$\frac{n}{n} = \frac{5^\circ}{7}$$

$$7n - 14 = 5n$$

$$2n = 14$$

$$n = 7$$

$\therefore$ it is a 7-sided figure.