UNIVERSITY OF WATERLOO
Faculty of Mathematics
Waterloo, Ontario N2L 3G1

Centre for Education in
Mathematics and Computing

Grade 6 Math Circles

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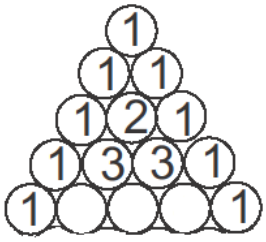
Pascal's Triangle

Pascal's Triangle

Pascal's Triangle is an interesting number pattern named after Blaise Pascal, a famous French mathematician. It has many uses in counting paths and it's use in the combination function will become really important to us for our next lesson.

Building the Pascal's Triangle

To build the triangle we start with 1 at the top, and continue adding numbers in a triangular shape. The leftmost and rightmost diagonals of Pascal's Triangle are 1s, and each number in between is the sum of the two numbers above it.



Rows and Elements

Pascal's Triangle has a unique classification method in order to identify rows and entries:

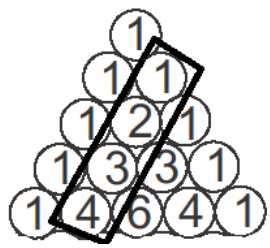
- A **row** refers to the horizontal set of numbers in the Pascal's Triangle. We count the very top "1" as being a part of Row 0. The counting scheme continues as normal and you can easily determine the row number by looking at the second number in the row (or the first non-1 value).
- An **entry** refers to a specific number in a said row. We call the very first number in a row as Entry 0 (which will always be the number 1). The counting scheme continues as normal, and the number of entries for a row are always between 0 and the row's number. For example: Row 3 {1331} has entry numbers 0, 1, 2, 3.

Interesting Patterns

The Pascal's Triangle is well-known because of the interesting counting patterns that can be found within it. We'll investigate some of the important ones:

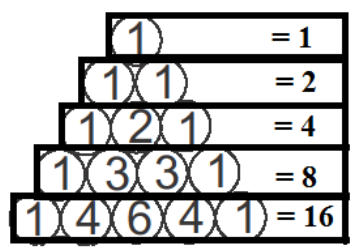
Counting Numbers

The boxed area covers all of the First Entries of each row. These are our standard counting numbers increasing by one each time. This is helpful because this number tells us which row we are working on.



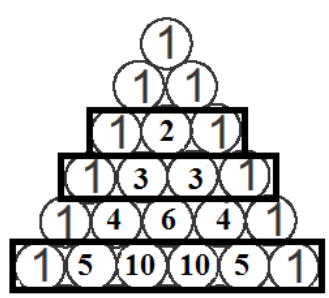
Sum of Rows

If we look at each row of Pascal's Triangle, we can see that the sum of each row is a power of 2. In fact, each row adds up to 2^n, where n is the row number.



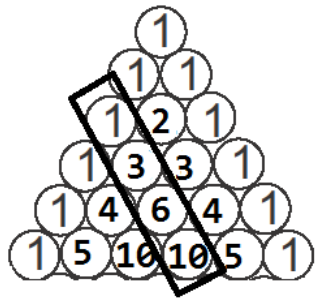
Prime Multiples

When we circle all the rows whose counting number is a prime number (**NOTE: Remember that a Prime Number is a number who is only divisible by one and itself**), we can see that every other entry in said row is a multiple of that prime number.

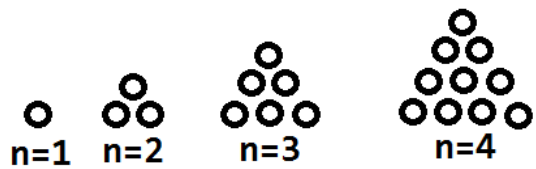


Triangular Numbers

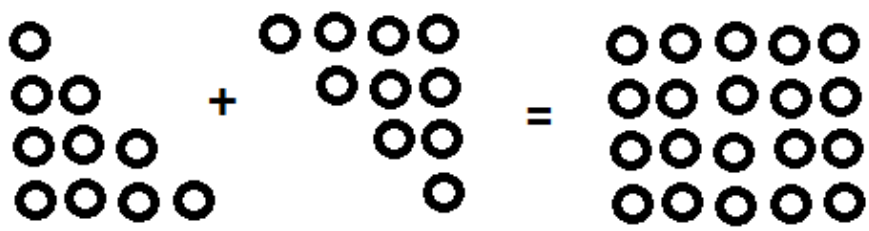
The third diagonal has the triangular numbers. Triangular numbers, are the numbers of dots that it takes to make increasingly large triangles.



If we draw out our Triangular Numbers in their triangle form we see how they correspond:



But how do we find the nth Triangular Number? If we take, for example, the 4th triangular number, and did not want to count all the dots that make it up, we could do the following:

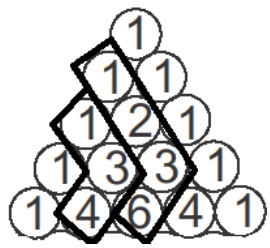


By adding two n=4 Triangles, we get a rectangle that is 5×4. This rectangle has 20 dots, and since we know the n=4 Triangle has 10 dots, we must divide by 2. This gives us the formula for any Triangular Number:

$$\frac{(n) \times (n+1)}{2}$$

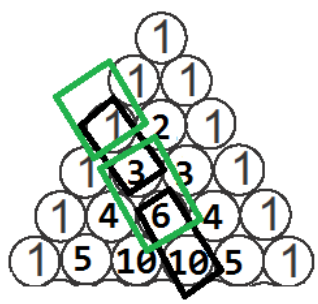
Hockey Sticks

Numbers selected on a diagonal starting at an outside one have a sum of the number below that is not on the same diagonal. Essentially, starting from a 1 on the diagonal, the numbers on the longer part of the hockey stick add up to the single number that makes up the "blade" of the hockey stick. You could make the longer part as long or as short and it will always work!



Square Numbers

A square number is the number resulting from multiplying a number by itself.



We can find the square numbers by adding up any two numbers in the triangular number diagonal.

For example:

- $0 + 1 = 1 = 1^2$
- $1 + 3 = 4 = 2^2$
- $3 + 6 = 9 = 3^2$
- $6 + 10 = 16 = 4^2$

11th Exponents

Take a row and find the sum of all its digits. When there is a multiple digit element place brackets around the preceding element's ones digit and the first digit of the multiple digit element. Then add the numbers inside the brackets together

Row Number	**Actual Row**	**Number**
0	1	$1=11^0$
1	1,1	11^1
2	1,2,1	$121=11^2$
3	1,3,3,1	$1331=11^3$
4	1,4,6,4,1	$14641=11^4$
5	1,5,10,10,5,1 1,(5+1),(0+1),0,5,1 1,6,1,0,5,1	$161051=11^5$
6	1,6,15,20,15,6,1 1,(6+1),(5+2),(0+1),5,6,1 1,7,7,1,5,6,1	$1771561=11^6$

If you add two numbers and it makes another multiple digit number, you must repeat the process with the new numbers.

Exercises I

1. Find the 8th and 9th row of Pascal's Triangle.
2. Which row has a sum of 2048?
3. Find the following triangular numbers:
   (a) 5th Triangular Number
   (b) 14th Triangular Number
   (c) 37th Triangular Number

Factorials

A common mathematical notation in counting is the **factorial notation**. The factorial of some positive integer, written as $n!$, is represented as:

$$(n) \times (n-1) \times (n-2) \times ... \times (2) \times (1)$$

Let's clear this up with an example. We can show that:

$$5! \;=\; 5 \times 4 \times 3 \times 2 \times 1 \;=\; 120$$

Essentially all we need to do is multiply every number from 1 to n to find n!. But factorials have some restrictions and special cases that we must look at:

- 0! is a special case that we must remember. We can say that $0! = 1$.
- We can only find the factorial of integers (whole numbers). For example 3.14! or $\pi!$ is not possible and therefore has no answer.
- We cannot find the factorial of negative integers. Therefore we cannot get an answer for $(-5)!$.

Multiplying & Dividing Factorials

While dividing factorials may seem like an easy task if you have a calculator on-hand, if your calculator does not have a factorial command, manually finding it by typing out expanded factorials can be a tedious task, especially with larger numbers. Learning to simplify factorials when we divide can make working with them much easier. Let's look at two cases:

Dividing: Numerator is Larger

After we have expanded all of the factorials in an expression, we can look for any common numbers. If the numerator is larger than the denominator we can see that there will be a product remaining in the numerator, but the denominator will be completely cancelled out.

Example: Simplify $\frac{6!}{3!}$

Solution: Without using our calculator and expanding the factorials we get:

$$\frac{6\times5\times4\times3\times2\times1}{3\times2\times1}$$

We can see that in both the numerator and denominator, there exists a "$3\times2\times1$". So let's cancel them out. This leaves us with:

$$6 \times 5 \times 4 = 120$$

Dividing: Denominator is Larger

After we have expanded all of the factorials in an expression, we can look for any common numbers. If the numerator is larger we can see that there will be a product remaining in the numerator.

If the denominator is larger than the numerator we can see that there will be a product remaining in the denominator, but the numerator will be completely cancelled out. If this occurs and nothing remains in the numerator, we must place a "1" in the numerator as a place-holder.

Example: Simplify $\frac{5!}{7!}$

Solution: Without using our calculator and expanding the factorials we get:

$$\frac{5\times4\times3\times2\times1}{7\times6\times5\times4\times3\times2\times1}$$

We can see that in both the numerator and denominator, there exists a "$5 \times 4 \times 3 \times 2 \times 1$". So let's cancel them out. Remember, because everything in the numerator is gone now, we have to place a "1" in the numerator. This leaves us with:

$$\frac{1}{7\times6} = \frac{1}{42}$$

Multiplying: Either Scenario

Multiplying factorials together does not introduce any options for us to simplify the expression. However, when we introduce multiplication together with division, it requires us to write out the expanded form of the factorials to assure that we cancel out appropriately. Remembering that larger parts of the factorials may further be factored allows for further simplifying of the expression.

Example: Simplify $\frac{6!\times 3!}{8!}$

Solution: Without using our calculator and expanding the factorials we get:

$$\frac{6\times5\times4\times3\times2\times1\times3\times2\times1}{8\times7\times6\times5\times4\times3\times2\times1}$$

We can see that in both the numerator and denominator, there exists a "$6 \times 5 \times 4 \times 3 \times 2 \times 1$". So let's cancel them out. This leaves us with:

$$\frac{3\times2\times1}{8\times7}$$

We may think this is as far as we can go. But we know that $8 = 2 \times 2 \times 2$. So let's insert this into our example:

$$\frac{3\times2\times1}{2\times2\times2\times7}$$

Let's cancel out a 2 from the numerator and denominator:

$$\frac{3\times1}{2\times2\times7} = \frac{3}{28}$$

Combinations

Combinations and the Triangle: Finding any Entry

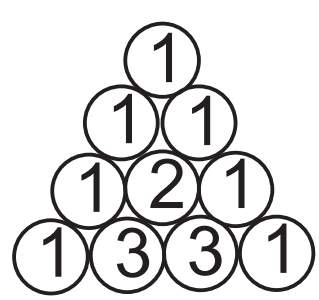
The formula for any entry in Pascal's Triangle can be found using the combination formula. The formula is:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

(where n is the row number, and k is the entry such that $0 \leq k \leq n$)

Let's see what this looks like when we compare the triangles:

$$\begin{array}{c} \binom{0}{0} \\ \binom{1}{0}\ \binom{1}{1} \\ \binom{2}{0}\ \binom{2}{1}\ \binom{2}{2} \\ \binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3} \end{array}$$



Example: Find the number for each entry:

(a) 6th entry of the 8th row.

$$\begin{aligned}\binom{8}{6} &= \frac{8!}{6!(8-6)!}\\ \binom{8}{6} &= \frac{8\times7\times6\times5\times4\times3\times2\times1}{6\times5\times4\times3\times2\times1\times2\times1}\\ \binom{8}{6} &= \frac{2\times2\times2\times7}{2\times1}\\ \binom{8}{6} &= 28\end{aligned}$$

(b) 12th entry of the 10th row.

We cannot find this because the 10th row only has entries from 0 to 10. It does not have a 12th entry.

The "Choose" Formula

When we have a set of items and want to know how many ways we can pick some of them, we write:

$${}_nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Example: At a cafeteria, a student is allowed to pick 4 items from the following list:

{ pop, juice, milk, water, burger, hotdog, vegetable soup, banana, orange, apple pie }

(a) How many ways can a student have a 4 piece meal?

(b) How many ways can a student have a 4 piece meal if they can only have one drink?

Solution: Let's investigate how many combinations exist:

(a) Using the formula, we have 10 items and want to choose 4:

$${}_{10}C_4 = \binom{10}{4} = \frac{10!}{4!(10-4)!} = \frac{10!}{4!(6)!} = \frac{10\times9\times8\times7}{4\times3\times2\times1} = \frac{5040}{24} = 210$$

There are 210 combinations that the student could have.

(b) This question follows the same idea except there are 4 drinks and he can only pick one, and then 6 other items from which he must pick the remaining 3:

$$\binom{4}{1}\times\binom{6}{3} = \frac{4}{1!3!}\times\frac{6}{3!3!} = 4\times20 = 80$$

Problem Set

1. Evaluate the following factorials:

   (a) $10!$

   (b) $\dfrac{15!}{13!}$

   (c) $\dfrac{4!}{9!}$

   (d) $\dfrac{5! \times 3!}{8! \times 2!}$

   (e) * $\dfrac{4! \times 6! \times 3!}{6! \times 2! \times 2! \times 5!}$

2. Find the missing number in this row. (**Hint:** Looking at the numbers is useful, but how many entries are there?)

   1 _ 78 186 715 1287 1716 1716 1287 715 186 78 _ 1

3. Carlos Pizza Shop offers 9 kinds of pizza toppings: onions, tomatoes, peppers, bacon, ham, extra cheese, spinach, pepperoni, olives.

   (a) How many kinds of pizza can someone order if they can only choose 1 topping?

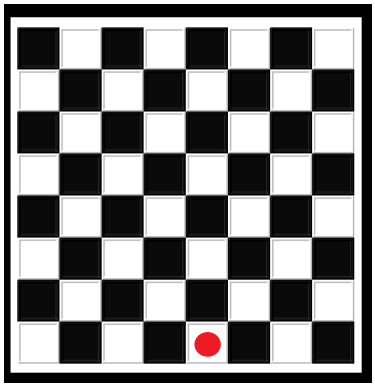
   (b) How many kinds of pizza can someone order if they can choose 8 toppings?

   (c) How many kinds of pizza can someone order if they can choose 3 toppings?

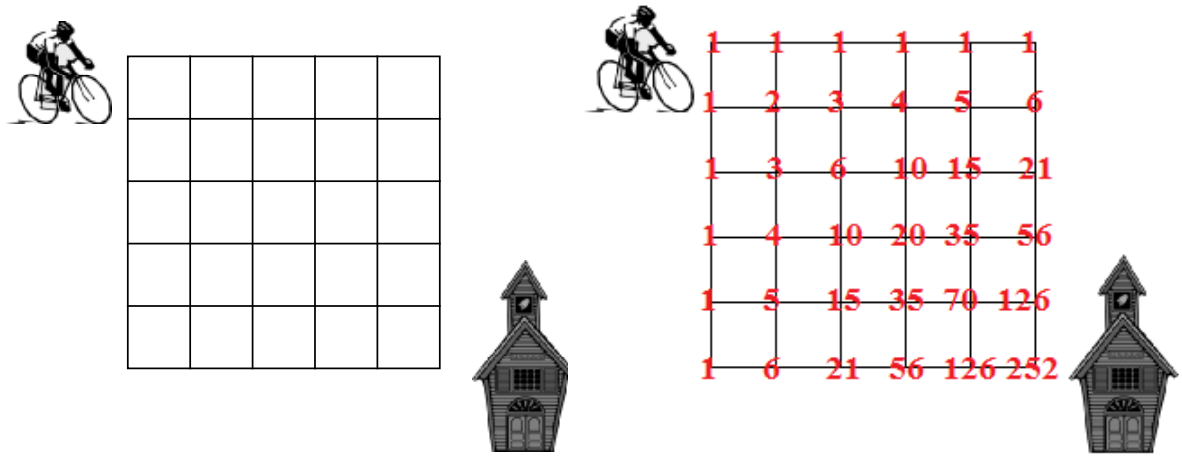
   (d) How many kinds of pizza can someone order if they can choose 7 toppings?

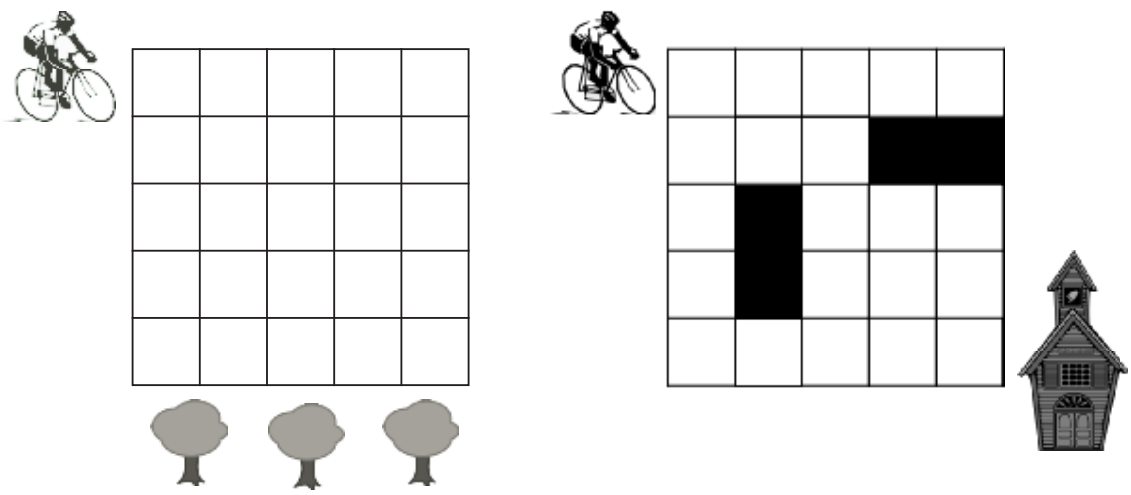
   (e) What do you notice about these values? Why do you think this is true?

4. . An eight-square checker board is shown below. A checker is placed at the third position in the bottom row. You are allowed to move the checker one square diagonally up (either left of right) at any one time. How many different ways are there to get to the opposite side of the board?



5. Count how many paths James can take to get to each destination if he can only travel down or to the right. For each example we count his paths at each corner of the squares. Here's an example:





Hint: In the left example, James considers the entire bottom line his destination. In the right example, James cannot cross blacked out squares.

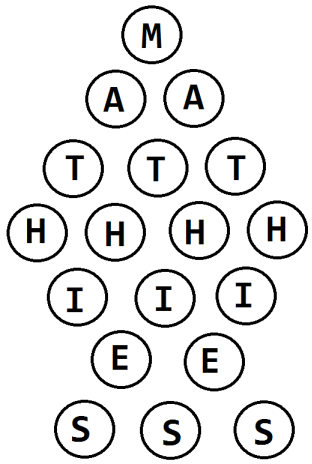
6. Evaluate the following:

   (a) $\binom{5}{3}$

   (b) $\binom{10}{6}$

   (c) $\dfrac{\binom{7}{6}}{\binom{4}{2}}$

7. How many ways can you spell MATHIES using the following Pascal triangle shape?



8. * Find 11^8 using the Pascal's Triangle method.

9. * Brad the Bachelor has dwindled down his choices to 10 lucky bachelorettes. How many combinations are there:

   (a) If the show is getting cancelled and he needs to pick 4 finalists?

   (b) If the show says he must pick Beautiful Betty and 3 other finalists?

   (c) If he doesn't want to pick Smelly Sandra and must pick 4 finalists.

10. * A school has 380 female students and 120 male students. They must create a 5-person student council.

    (a) In how many ways can they do this?

    (b) In how many ways can they do this if it can only be made of girls?

    (c) In how many ways can they do this if there must be 4 boys and 1 girl?

    (d) In how many ways can they do this if there must be more girls than boys?

11. ** The following is a portion of Pascal's Triangle. Find the values of X, Y and Z:

$$\begin{array}{ccccccc} 1287 & & \underline{\quad\quad} & & \underline{\quad\quad} & & X \\ & 3003 & & 3432 & & Y & \\ & & \underline{\quad\quad} & & 6435 & & \\ & & & Z & & & \end{array}$$

12. **** Show that $\binom{12}{2} + \binom{12}{3} = \binom{13}{3}$ by using the choose formula to re-write the equation.