Math Circles: Diophantine Equations II
(Geometric interpretation of linear Diophantine equations)

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2. $6x + 4y = 2$
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Consider the following linear Diophantine equations:

1. \[ 6x + 4y = 5 \]

Since \( \gcd(6, 4) = 2 \), and 2 * does not divide * 5, this equation does not have an integer solution.

2. \[ 6x + 4y = 2 \]

Since \( \gcd(6, 4) = 2 \), and 2 * does divide * 2, this equation does have an integer solution. We can find one by inspection: \( x = 1, y = -1 \). In fact, there are infinitely many integer solutions.

Let’s examine these two situations geometrically...
The equation $6x + 4y = 5$

Notice that this is the equation of a line. We can rearrange the equation to get:

$$
6x + 4y = 5 \\
4y = -6x + 5 \\
y = -\frac{3}{2}x + \frac{5}{4}
$$

Since this is a line, there are infinitely many pairs of numbers $x$ and $y$ that satisfy $y = -\frac{3}{2}x + \frac{5}{4}$, but we know that there are no integer pairs, because the equation $6x + 4y = 5$ has no integer solutions.

On a graph...
The equation $6x + 4y = 5$

Lattice points are points in the plane with integer $x$ and $y$ coordinates.

These will correspond to integer solutions of our linear Diophantine equation.

The line misses all the lattice points shown in this graph.
The equation $6x + 4y = 5$

The line misses all lattice points.

The equation has no integer solutions.
The equation $6x + 4y = 2$

Again, this is the equation of a line. We can rearrange the equation to get:

\[
6x + 4y = 2 \\
4y = -6x + 2 \\
y = -\frac{6}{4}x + \frac{2}{4} \\
y = \frac{3}{2}x + \frac{1}{2}
\]

This second line has the same slope as the first line, but with a different $y$ intercept.

Since this is a line, there are infinitely many pairs of numbers $x$ and $y$ that satisfy $y = -\frac{3}{2}x + \frac{1}{2}$. Let’s convince ourselves with a picture that there are also infinitely many integer solutions.
The equation $6x + 4y = 2$

$y = -\frac{3}{2}x + \frac{1}{2}$

The line hits many lattice points on this graph.

$(-3, 5)$

$(-1, 2)$

$(1, -1)$

$(3, -4)$
The equation $6x + 4y = 2$

The line hits many lattice points on this graph.

$(-3, 5)$

$(-1, 2)$

$(1, -1)$

$(3, -4)$
The equation $6x + 4y = 2$.

If we pick one lattice point, then we can find its “neighbouring lattice points” as follows.
The equation $6x + 4y = 2$

If we pick one lattice point, then we can find its “neighbouring lattice points” as follows.

$$(1 - 2, -1 + 3) = (-1, 2)$$

$$\uparrow$$

$$(1, -1)$$

$$\downarrow$$

$$(1 + 2, -1 - 3) = (3, -4)$$
Given one solution \( x_0, y_0 \) to an equation \( ax + by = c \), how do we find its “neighbouring solutions”?

Assuming that \( a, b \neq 0 \), let’s rearrange this equation as follows:

\[
\begin{align*}
ax + by &= c \\
by &= -ax + c \\
y &= -\frac{a}{b}x + \frac{c}{b}
\end{align*}
\]

giving us a slope of \( m = -\frac{a}{b} \).

So we might be tempted so say “move \( a \) units up and \( b \) units to the left” or “move \( b \) units to the right and \( a \) units down”.

This will indeed find us another solution, but which one?
Back to our example

Let’s return to the example $6x + 4y = 2$. Then we have

$$y = -\frac{3}{2}x + \frac{1}{2}$$

with $a = 6$ and $b = 4$. 
Back to our example

Let’s return to the example $6x + 4y = 2$. Then we have

$$y = -\frac{3}{2}x + \frac{1}{2}$$

with $a = 6$ and $b = 4$.

If we simply move $a = 6$ units up and $b = 4$ units to the left then we may miss lattice points!
Neighbouring solutions

\[ ax + by = c \implies y = -\frac{a}{b}x + \frac{c}{b} \]

While \(-\frac{a}{b}\) is the slope of the line, in order to use this fraction to find ALL lattice points, we need to make sure that we consider the fraction \(\frac{a}{b}\) in lowest terms.

The fraction \(\frac{a}{b}\) in lowest terms = \(\frac{a}{\gcd(a, b)}\) \(\frac{b}{\gcd(a, b)}\)

To find the “neighbouring solutions”:

- Move right \(\frac{b}{\gcd(a, b)}\) units, and down \(\frac{a}{\gcd(a, b)}\) units, or
- Move up \(\frac{a}{\gcd(a, b)}\) units, and left \(\frac{b}{\gcd(a, b)}\) units.
Example

Given that $x_0 = 1$, $y_0 = -1$ is one solution to the equation $6x + 4y = 2$, find the two “neighbouring” integer solutions.

Solution: Using the method from the previous slide:

First we find $\frac{a}{\gcd(a, b)} = \frac{6}{2} = 3$ and $\frac{b}{\gcd(a, b)} = \frac{4}{2} = 2$.

To find the “neighbouring” solutions we do the following:

- $x = x_0 + \frac{b}{\gcd(a, b)} = 1 + 2 = 3$ “right 2 units”
  
- $y = y_0 - \frac{a}{\gcd(a, b)} = -1 - 3 = -4$ “down 3 units”

- $x = x_0 - \frac{b}{\gcd(a, b)} = 1 - 2 = -1$ “left 2 units”
  
- $y = y_0 + \frac{a}{\gcd(a, b)} = -1 + 3 = 2$ “up 3 units”
Given one solution $x_0 = 1$, $y_0 = -1$, the "next lattice points" are:

$(1 - 2, -1 + 3) = (-1, 2)$

$(1 + 2, -1 - 3) = (3, -4)$
Example

Given that $x_0 = 9$, $y_0 = -20$ is one integer solution to the equation $483x + 217y = 7$, find the two “neighbouring” integer solutions.

**Solution:** We have $a = 483$ and $b = 217$ and $\gcd(483, 217) = 7$ from earlier calculations.

$$\frac{a}{\gcd(a, b)} = \frac{483}{7} = 69 \quad \text{and} \quad \frac{b}{\gcd(a, b)} = \frac{217}{7} = 31.$$

To find the “neighbouring” solutions we do the following:

- $x = x_0 + \frac{b}{\gcd(a, b)} = 9 + 31 = 40$ “right 31 units”
- $y = y_0 - \frac{a}{\gcd(a, b)} = -20 - 69 = -89$ “down 69 units”
- $x = x_0 - \frac{b}{\gcd(a, b)} = 9 - 31 = -22$ “left 31 units”
- $y = y_0 + \frac{a}{\gcd(a, b)} = -20 + 69 = 49$ “up 69 units”

**Check!** $483(40) + 217(-89) = 7$ and $483(-22) + 217(49) = 7$. 
Finding all solutions

Suppose that \( x_0, y_0 \) is one integer solution to the linear Diophantine equation \( ax + by = c \), and let \( d = \gcd(a, b) \). Then the full set of integer solutions for the equation is given by

\[
x = x_0 + n \left( \frac{b}{d} \right), \quad y = y_0 - n \left( \frac{a}{d} \right)
\]

where \( n \) is any integer.

Note that \( n = 1 \) corresponds to a “neighbour” of \( x_0, y_0 \):

\[
x = x_0 + 1 \left( \frac{b}{d} \right) \quad y = y_0 - 1 \left( \frac{a}{d} \right)
\]
\[
= x_0 + \frac{b}{d} \quad = y_0 - \frac{a}{d}
\]

Also \( n = -1 \) corresponds to the other “neighbour” of \( x_0, y_0 \):

\[
x = x_0 + (-1) \left( \frac{b}{d} \right) \quad y = y_0 - (-1) \left( \frac{a}{d} \right)
\]
\[
= x_0 - \frac{b}{d} \quad = y_0 + \frac{a}{d}
\]
Find ALL integer solutions to the equation $6x + 4y = 2$.

**Solution:**
We know that $a = 6$, $b = 4$ and $\gcd(a, b) = \gcd(6, 4) = 2 = d$. Since $x_0 = 1$, $y_0 = -1$ is one solution to the given equation, the full set of integer solutions is given by

$$
x = x_0 + n \left( \frac{b}{d} \right) \quad y = y_0 - n \left( \frac{a}{d} \right)
$$

$$
= 1 + n \left( \frac{4}{2} \right) \quad = -1 - n \left( \frac{6}{2} \right)
$$

$$
= 1 + 2n \quad = -1 - 3n
$$

1. Every integer $n$ produces a particular solution $x = 1 + 2n$, $y = -1 - 3n$ to the equation, and
2. Every integer solution to the equation is of the form $x = 1 + 2n, y = -1 - 3n$ for some integer $n$. 
General solution \( x = 1 + 2n, \ y = -1 - 3n \)

\[
y = -\frac{3}{2}x + \frac{1}{2}
\]

\( n = -3 \)

\( n = -2 \)

\( n = -1 \)

\( n = 0 \) (original solution)

\( n = 1 \)

\( n = 2 \)

\( n = 3 \)