UNIVERSITY OF WATERLOO
Faculty of Mathematics
Waterloo, Ontario N2L 3G1

Senior Math Circles

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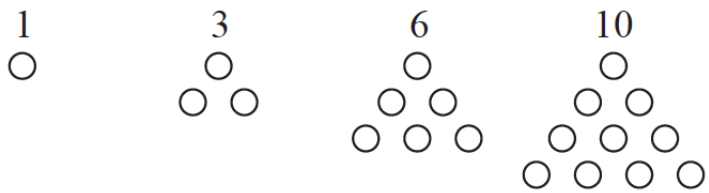
Sequences and Series I

1 Polygonal numbers

Consider the familiar sequence of *triangular numbers* $1, 3, 6, 10, 15, \ldots$ obtained by arranging circles in an equilateral triangle:



Question 1.1. What is the formula for the n-th term of this sequence?

Solution. Let u_n denote the n-th triangular number. Then

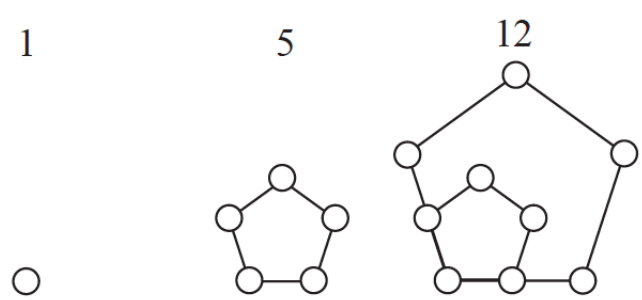
$$u_n = 1 + 2 + \cdots + (n-1) + n = \begin{cases} (1+n) + (2 + (n-1)) + \cdots + \left(\frac{n-1}{2} + \frac{n+1}{2}\right) & n \text{ is odd,} \\ (1+n) + (2 + (n-1)) + \cdots + \left(\frac{n-2}{2} + \frac{n+2}{2}\right) + \frac{n}{2} & n \text{ is even.} \end{cases}$$

In both cases, we find that

$$u_n = \frac{n(n+1)}{2}. \qquad \square$$

In this case, we were able to use special properties of triangular numbers to derive a formula for the n-th triangular number. We would like to find a more general method for finding such formulas without relying on any special properties. One general method that works in many cases is the method of *undetermined coefficients*, which we describe using the following example.

Example 1.2. The sequence $1, 5, 12, 22, 35, \ldots$ of *pentagonal numbers* consists of the number of objects obtained by arranging circular disks into layers of regular pentagons, as follows:



Question 1.3. What is the formula for the n-th pentagonal number?

Solution. Let u_n denote the n-th pentagonal number. We assume that u_n is a quadratic in n. This assumption is reasonable, since the size of a 2-dimensional object in general grows quadratically in n.

Write $u_n = an^2 + bn + c$. Then

$$\begin{aligned} u_1 &= a + b + c = 1 \\ u_2 &= 4a + 2b + c = 5 \\ u_3 &= 9a + 3b + c = 12 \end{aligned}$$

We can solve this system of equations by subtracting the first from the second, and the second from the third, to obtain

$$\begin{aligned} 3a + b &= 4 \\ 5a + b &= 7 \end{aligned}$$

Subtracting the first equation above from the second, we obtain $2a = 3$, so $a = 3/2$, $b = -1/2$, and $c = 0$. Hence

$$u_n = \frac{3}{2}n^2 - \frac{1}{2}n = \frac{3n^2 - n}{2}.$$

□

These methods also work for three-dimensional polygonal numbers:

Question 1.4. What is the formula for the n-th tetrahedral number?

Solution. Let u_n denote the n-th tetrahedral number. Since tetrahedrons are 3-dimensional objects, we assume that u_n is a cubic polynomial in n. Write $u_n = an^3+bn^2+cn+d$. The first few tetrahedral numbers are $1, 4, 10, 20$, so we have

$$\begin{aligned} u_1 &= a + b + c + d = 1 \\ u_2 &= 8a + 4b + 2c + d = 4 \\ u_3 &= 27a + 9b + 3c + d = 10 \\ u_4 &= 64a + 16b + 4c + d = 20 \end{aligned}$$

Subtracting consecutive equations yields

$$\begin{aligned} 7a + 3b + c &= 3 \\ 19a + 5b + c &= 6 \\ 37a + 7b + c &= 10 \end{aligned}$$

Subtracting consecutive equations again yields

$$\begin{aligned} 12a + 2b &= 3 \\ 18a + 2b &= 4 \end{aligned}$$

so $a = 1/6$, $b = 1/2$, $c = 1/3$, $d = 0$. Hence

$$u_n = \frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n = \frac{n(n+1)(n+2)}{6}.$$

□

2 Finite differences

The method of undetermined coefficients requires assuming that the general form of the formula (except for the unknown coefficients) is known in advance. An alternative method, called the method of *finite differences*, enables the formula to be computed without knowing the general form of the formula in advance.

The method of finite differences operates as follows. Let $u_0, u_1, u_2, \ldots$ be any sequence, and consider the sequence $\Delta u_0, \Delta u_1, \Delta u_2, \ldots$ defined by

$$\begin{aligned}
\Delta u_0 &= u_1 - u_0 \\
\Delta u_1 &= u_2 - u_1 \\
\Delta u_2 &= u_3 - u_2 \\
\Delta u_3 &= u_4 - u_3 \\
&\vdots
\end{aligned}$$

Note that u_n is a partial sum of the sequence $u_0, \Delta u_0, \Delta u_1, \Delta u_2, \ldots$ in the sense that

$$u_n = u_0 + (\Delta u_0 + \Delta u_1 + \cdots + \Delta u_{n-1}).$$

Now observe that

$$\begin{aligned}
u_1 &= u_0 + \Delta u_0 \\
u_2 &= u_1 + \Delta u_1 \\
u_3 &= u_2 + \Delta u_2 \\
u_4 &= u_3 + \Delta u_3
\end{aligned}$$

and in general $u_{n+1} = u_n + \Delta u_n$. If we take a leap of faith and consider Δ as an algebraic object, we may "factor" out the symbol Δ to obtain the formula

$$u_{n+1} = (1 + \Delta)u_n,$$

valid for any positive integer n.

We now proceed one step further. Define

$$\begin{aligned}
\Delta^2 u_0 &= \Delta u_1 - \Delta u_0 \\
\Delta^2 u_1 &= \Delta u_2 - \Delta u_1 \\
\Delta^2 u_2 &= \Delta u_3 - \Delta u_2 \\
\Delta^2 u_3 &= \Delta u_4 - \Delta u_3 \\
&\vdots
\end{aligned}$$

A picture may be helpful here. If we arrange the terms in rows according to the diagram

$$\begin{array}{ccccccccccccccc}
u_0 & & u_1 & & u_2 & & u_3 & & u_4 & & u_5 & & u_6 & & \cdots \\
& \Delta u_0 & & \Delta u_1 & & \Delta u_2 & & \Delta u_3 & & \Delta u_4 & & \Delta u_5 & & \cdots & \\
& & \Delta^2 u_0 & & \Delta^2 u_1 & & \Delta^2 u_2 & & \Delta^2 u_3 & & \Delta^2 u_4 & & \cdots & &
\end{array}$$

then each term is just the difference of the two terms lying above it. We now have

$$\begin{aligned} u_2 &= (1+\Delta)u_1 = u_1 + \Delta u_1 = (u_0 + \Delta u_0) + \Delta u_1 \\ &= (u_0 + \Delta u_0) + (\Delta u_0 + \Delta^2 u_0) = u_0 + 2\Delta u_0 + \Delta^2 u_0. \end{aligned}$$

If we again treat Δ as an algebraic object and factor it out, then we obtain

$$\begin{aligned} u_2 &= (1+\Delta)^2 u_0 \\ u_3 &= (1+\Delta)^2 u_1 \\ u_4 &= (1+\Delta)^2 u_2 \\ u_5 &= (1+\Delta)^2 u_3 \\ &\vdots \end{aligned}$$

In general, set

$$\Delta^{k+1} u_n = \Delta^k u_{n+1} - \Delta^k u_n.$$

Then, for any n and k, we have

$$u_{n+k} = (1+\Delta)^k u_n,$$

and in particular

$$u_n = (1+\Delta)^n u_0$$

for all n. This formula can be viewed as a generalization of the binomial theorem and Pascal's triangle to arbitrary sequences.

Question 2.1. What is the formula for the n-th r-gonal number?

Solution. We determine by inspection that the first few r-gonal numbers u_n, for any r, are:

$$\begin{aligned} u_0 &= 0 \\ u_1 &= 1 \\ u_2 &= r \\ u_3 &= 3r - 3 \\ u_4 &= 6r - 8 \\ u_5 &= 10r - 15 \end{aligned}$$

We form the following table of differences.

$$\begin{array}{rccccccccccccc} u_n: & 0 & & 1 & & r & & 3r-3 & & 6r-8 & & 10r-15 & & \cdots \\ \Delta u_n: & & 1 & & r-1 & & 2r-3 & & 3r-5 & & 4r-7 & & \cdots & \\ \Delta^2 u_n: & & & r-2 & & r-2 & & r-2 & & r-2 & & \cdots & & \\ \Delta^3 u_n: & & & & 0 & & 0 & & 0 & & \cdots & & & \end{array}$$

Using the binomial theorem, we conclude

$$\begin{aligned} u_n &= (1+\Delta)^n u_0 = u_0 + n\Delta u_0 + \binom{n}{2}\Delta^2 u_0 + \binom{n}{3}\Delta^3 u_0 + \cdots \\ &= n + \frac{n(n-1)}{2}(r-2) + 0 + \cdots = n + \frac{n(n-1)(r-2)}{2}. \end{aligned}$$

□