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Enriching Mathematics and
Computer Science for 50 years

Senior Math Circles

October 9, 2013

Problem Solving, Part 1 - Problem Set

1. The lines $ax + y = 30$ and $x + ay = k$ intersect at the point $P(6, 12)$. Determine the value of k.

2. The sum of the first n terms of a sequence is $S_n = 3^n - 1$, where n is a positive integer.

   (a) If t_n represents the nth term of the sequence, determine t_1, t_2, t_3.

   (b) Prove that $\frac{t_{n+1}}{t_n}$ is constant for all values of n.

3. The numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$ are written down individually on nine slips of paper. These nine slips of paper are placed into a hat, and then withdrawn, at random, without replacement, and their digits are recorded in order to form a four-digit integer N. What is the probability that N is divisible by 99?

4. Suppose you have a music player that requires two charged batteries to operate: that is, putting in two charged batteries will cause it turn on, and putting in one uncharged battery with any other battery (charged or uncharged) will cause the player to remain off. You have a pile of k batteries, of which you know that some g of them are good. You would like to figure out the minimum number of attempts to make the music player turn on. (An attempt consists of putting two batteries into the music player and checking if it is on.)

   (a) Show how you can turn on the music player using 3 attempts if you have three batteries, one of which is uncharged.

   (b) Show how you can turn on the music player using 3 attempts if you have six batteries, two of which are uncharged.

   (c) Show how you can turn on the music player using 7 attempts if you have eight batteries, four of which are uncharged.

5. A game for two players uses four counters on a board which consists of a 20×1 rectangle. The two players alternate turns. A turn consists of moving any one of the four counters any number of squares to the right, but the counter may not land on top of, or move past, any of the other counters. For instance, in the position shown below, the next player could move D one, two or three squares to the right, or move C one or two squares to the right, and so on.

The winner of the game is the player who makes the last legal move. (After this move the counters will occupy the four squares on the extreme right of the board and no further legal moves will be possible.) In the position shown above, it is your turn. Which move should you make and what should be your strategy in subsequent moves to ensure that you will win the game?