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Grade 6 Math Circles

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Dimensional and Unit Analysis - Solutions

1. See charts. Remember to reference your charts :)

2. The answer is no, not any seven dimensions can be base dimensions.

   Now to answer this question with mathematical certainty is very difficult and beyond what I expect you to know how to do. However, an easy way to think about this is to consider a group of seven dimensions and show that they cannot be used to derive a dimension that you know.

   For example, if we replace mass with the dimension speed, no matter which of the other dimensions you multiply or divide together, you will not be able to derive the dimension of mass from these (try to think about it physically). So there is one example of a group of seven dimensions that cannot be base dimensions. Therefore, not any seven dimensions can be base dimensions.

   Note that there are some other combinations of seven dimensions that do work though. For example, if you replace mass with density, then you can easily derive mass from length and density, and therefore you can derive all the dimensions that depend on mass.

3. (a) $4^{-3} = \frac{1}{4^3} = \frac{1}{(4)(4)(4)} = \frac{1}{64}$

   (b) $10^{-7} = \frac{1}{10^7} = \frac{1}{10000000} = 0.0000001$

   (c) $0.0001 = \frac{1}{10000} = \frac{1}{10^4} = 10^{-4}$

4. (a) From the prefix chart, we can see that 1 ng $= 10^{-9}$ g $= 0.000000001$ g

(b) First, we convert to the zero-order unit of seconds.

From the prefix chart, we can see that 1 ks $= 10^3$ s $= 1000$ s. Therefore, we can create the unit identity

$$\frac{1000 \text{ s}}{1 \text{ ks}} = 1$$

Thus,

$$5 \text{ ks} = (5 \text{ ks})\left(\frac{1000 \text{ s}}{1 \text{ ks}}\right) = (5 \cancel{\text{ks}})\left(\frac{1000 \text{ s}}{1 \cancel{\text{ks}}}\right) = 5000 \text{ s}$$

Next we can convert to the desired units of milliseconds.

Also from the prefix chart, we know that 1 ms $= 10^{-3}$ s $= 0.001$ s. Or we could write this as 1000 ms $=$ 1 s (since one millisecond is one one-thousandth of a second). Therefore, we can create the unit identity

$$\frac{1000 \text{ ms}}{1 \text{ s}} = 1$$

Thus,

$$5000 \text{ s} = (5000 \text{ s})\left(\frac{1000 \text{ ms}}{1 \text{ s}}\right) = (5000 \cancel{\text{s}})\left(\frac{1000 \text{ ms}}{1 \cancel{\text{s}}}\right) = 5,000,000 \text{ ms}$$

So in conclusion, 5 ks $= 5,000,000$ ms.

(c) From the prefix table, 1 MJ $= 10^6$ J $= 1000000$ J, 1 GJ $= 10^9$ J $= 1000000000$ J. Then we can create the unit identities

$$\frac{1000000 \text{ J}}{1 \text{ MJ}} = \frac{1 \text{ GJ}}{1000000000 \text{ J}} = 1$$

Thus,

$$1000 \text{ MJ} = (1000 \cancel{\text{MJ}})\left(\frac{1000000 \cancel{\text{J}}}{1 \cancel{\text{MJ}}}\right)\left(\frac{1 \text{ GJ}}{1000000000 \cancel{\text{J}}}\right) = 1 \text{ GJ}$$

(d) From the prefix table, 1 hK $= 10^2$ K $= 100$ K, 1 daK $= 10^1$ K $= 10$ K. Then we can create the unit identities

$$\frac{100 \text{ K}}{1 \text{ hK}} = \frac{1 \text{ daK}}{10 \text{ K}} = 1$$

Thus,

$$10 \text{ hK} = (\cancel{10} \cancel{\text{hK}})\left(\frac{100 \cancel{\text{K}}}{1 \cancel{\text{hK}}}\right)\left(\frac{1 \text{ daK}}{\cancel{10} \cancel{\text{K}}}\right) = 100 \text{ daK}$$

(e) First, we should convert 1 m to centimetres.
From the prefix chart, 1 cm $= 10^{-2}$ m $= 0.01$ m. Or we can write 100 cm $= 1$ m.
Therefore,

$$1 \text{ m}^3 = (1 \text{ m})(1 \text{ m})(1 \text{ m}) = (100 \text{ cm})(100 \text{ cm})(100 \text{ cm}) = 1,000,000 \text{ cm}^3$$

5. A leap year has 366 days. We will use unit identities to convert the units of time from days to seconds.
The following is true:

$$1 \text{ day} = 24 \text{ h} \qquad 1 \text{ h} = 60 \text{ min} \qquad 1 \text{ min} = 60 \text{ s}$$

Therefore, we can create the following unit identities:

$$\frac{24 \text{ h}}{1 \text{ day}} = \frac{60 \text{ min}}{1 \text{ h}} = \frac{60 \text{ s}}{1 \text{ min}} = 1$$

Thus,

$$366 \text{ day} = (366 \cancel{\text{day}}) \left(\frac{24 \cancel{\text{h}}}{1 \cancel{\text{day}}}\right) \left(\frac{60 \cancel{\text{min}}}{1 \cancel{\text{h}}}\right) \left(\frac{60 \text{ s}}{1 \cancel{\text{min}}}\right) = (366)(24)(60)(60) \text{ s} = 31,622,400 \text{ s}$$

So there are $31,622,400$ seconds in one leap year.

6. Since we are given the density of the secret ingredient, if we can find the volume of the secret ingredient in Bob's cider then we can calculate the mass of the secret ingredient in Bob's cider by the formula for density:

$$\text{density} = \frac{\text{mass}}{\text{volume}} \qquad \Rightarrow \qquad \text{mass} = (\text{volume})(\text{density})$$

Since the secret ingredient makes up 1% of the cider (by volume), the volume of the secret ingredient in Bob's cider is

$$V = (0.01)(3 \text{ L}) = 0.03 \text{ L}$$

Since the density is given in g/mL and we want to express the mass of the secret ingredient in grams, it makes sense to convert V to mL.
From the prefix chart, 1 mL $= 10^{-3}$ L $= 0.001$ L. Or we can write, 1000 mL $= 1$ L.

Therefore,

$$V = 0.03 \text{ L} = (0.03 \cancel{\text{L}}) \left(\frac{1000 \text{ mL}}{1 \cancel{\text{L}}} \right) = 30 \text{ mL}$$

Now we can use the density formula to calculate the mass, m, of the secret ingredient in Bob's 3 L of cider:

$$m = (30 \cancel{\text{mL}}) \left(\frac{0.5 \text{ g}}{1 \cancel{\text{mL}}} \right) = 15 \text{ g}$$

So there are 15 grams of the secret ingredient in Bob's 3 litres of his grandmother's cider.

7. To solve the question, we really have to solve two smaller questions:

   (1) What is the minimum cost of giving 20 people one pop each?

   (2) What is the minimum cost of giving 20 people 3 slices of pizza each?

   Let's solve question (1) first:
   Since each person wants one pop, Jessica must buy at least 20 pops to ensure everyone at the party gets a pop.
   We can think of a *case* as a unit and a *pop* as a unit. Then 1 case = 12 pop. Clearly, one case of pop is not enough since $12 < 20$. However, 2 cases is enough:

$$2 \text{ case} = (2 \cancel{\text{case}}) \left(\frac{12 \text{ pop}}{1 \cancel{\text{case}}} \right) = 24 \text{ pop}$$

   The cost of 2 cases of pop is

$$(2 \cancel{\text{case}}) \left(\frac{\$10}{1 \cancel{\text{case}}} \right) = \$20$$

   If Jessica buys 20 cans of pop instead, it will cost her

$$(20 \cancel{\text{pop}}) \left(\frac{\$1.25}{1 \cancel{\text{pop}}} \right) = \$25$$

   Therefore, the minimum amount of money Jessica can spend on pop is \$20.

   Now to solve question (2):
   Now, we will begin to think of person as a unit of people, and large, xlarge, and slice as units of pizza as well. Since there are 10 slices in a large pizza, 1 large = 10 slice. Similarly, 1 xlarge = 12 slice.

The total amount of pizza Jessica needs to buy is

$$(20\ \cancel{\text{person}})\left(\frac{3\ \text{slice}}{1\ \cancel{\text{person}}}\right) = 60\ \text{slice}$$

Now using "unit identities" (we're not using real units), we can get that

$$60\ \text{slice} = (60\ \cancel{\text{slice}})\left(\frac{1\ \text{large}}{10\ \cancel{\text{slice}}}\right) = 6\ \text{large}$$

and that

$$60\ \text{slice} = (60\ \cancel{\text{slice}})\left(\frac{1\ \text{xlarge}}{12\ \cancel{\text{slice}}}\right) = 5\ \text{xlarge}$$

That is, Jessica can either buy 6 large pizzas or 5 extra large pizzas to make sure that everybody has 3 slices of pizza.
The cost of 6 large pizzas is

$$(6\ \cancel{\text{large}})\left(\frac{\$12}{1\ \cancel{\text{large}}}\right) = \$72$$

The cost of 5 extra large pizzas is

$$(5\ \cancel{\text{xlarge}})\left(\frac{\$14}{1\ \cancel{\text{xlarge}}}\right) = \$70$$

Clearly $70 < 72$, so the minimum amount of money Jessica can spend on pizza is \$70.

Putting the answers of (1) and (2) together, we get that the minimum amount of money that Jessica can spend (so that everyone gets their share of pizza and pop) is $\$20 + \$70 = \$90$.

This question isn't really a dimensional or unit analysis question. However, I included this question because we can use the ideas of cancelling, ratios, and units to help us solve the question.

Aside: An Interesting Note

In this solution, I made up "units" like *person* or *slice*. Really, all of these quantities can be argued to have the dimension of *amount of substance*, N. But aren't we just counting? Shouldn't all these values really be dimensionless? This is a subtle point, but there is a difference between the ideas of N and a dimensionless quantity. When a number or variable has the dimension N, we know that the number or variable corresponds to a counted number of *physical objects* of some substance. This substance

could be oxygen, or it could be pizza. Meanwhile, dimensionless quantities are pure numbers. Dimensionless counting is like counting without objects (it is an abstract idea). To illustrate the difference between N and dimensionless numbers, consider multiplying a length by 2 versus multiplying the length by 2 pizzas (do you see the difference?). If you would like to discuss these ideas some more, please speak to me in class.

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8. In expression (a), $[x+y] = [x] = [y]$ by our first rule for dimensional quantities. But we are given that $[x] = [y] \neq [z]$. Therefore, $[x+y] \neq [z]$. And so by the first rule again, the expression $x + y = z$ does not make sense. That is, we cannot equate quantities of different dimensions.

   In expression (b), x and y are said to be quantities of the same dimension. Therefore, by the first rule for dimensional quantities, we may subtract y from x. So expression (b) makes sense.

   In expression (c), x, y, z are all quantities of different dimensions. However, the expression only contains the operations of multiplication and division with the variables. Therefore, the second rule for dimensional quantities tells us that expression (c) is acceptable and makes sense.

   The statement (d) is comparing two quantities of different dimensions (length and mass). This comparison is not allowed by the first rule for dimensional quantities. Thus, the statement does not make sense.

   Statement (e) compares the ratio of length to mass of each fence A and B. The ratio of length to mass for each fence is a quantity of dimension L/M. The division of the two quantities of different dimensions is allowed by the second rule for dimensional quantities. Now we know that statement (e) is comparing two quantities of the same dimension L/M. This comparison is allowed by the first rule for dimensional quantities. Therefore, statement (e) makes sense.

   Thus, the only expressions/statements that do not make sense are (a) and (d).

9. Let m be the mass contained in V. Then the formula for density is

$$D = \frac{m}{V}$$

Therefore,

$$[D] = \frac{[m]}{[V]} = \frac{M}{L^3} \tag{1}$$

But the formula for speed is

$$S = \frac{x}{t}$$

where x is some distance travelled in a time t. Thus,

$$[S] = \frac{[x]}{[t]} = \frac{L}{T} \tag{2}$$

Multiplying both sides of equation (2) by T gives

$$L = [S]T$$

So we can substitute this expression for L into equation (1) to get

$$[D] = \frac{M}{([S]T)^3}$$

It is unnatural to define density in terms of these dimensions because they are not base dimensions. As well as being special dimensions mathematically, base dimensions are often seem more natural because they correspond to physical ideas that we are very familiar with.

10. Since r is the radius of the base of the cone and s is the slant height of the cone, then we have that $[r] = [s] = L$. Also, note that π is dimensionless.
Then by the three rules for dimensional quantities, we can see that

$$[\pi r^2 + \pi rs] = [r^2] = [rs] = [r]^2 = [r][s] = L^2$$

Now, realize that the dimension of volume is L^3 and that the dimension of surface area is L^2. So if the expression $\pi r^2 + \pi rs$ must be a formula for either the volume or the surface area of a cone, then it must be the formula for the surface area of a cone.

11. (a) From the base unit chart, the kilogram is the base unit for the base dimension M. Similarly, we can find that the metre is the base unit for the dimension L and that the second is the base unit for the dimension T.

Since dimensions must follow the same relationship that their units do, we can realize that

$$\mathrm{N} = \frac{(\mathrm{kg})(\mathrm{m})}{\mathrm{s}^2} \qquad \Rightarrow \qquad [F] = \frac{ML}{T^2}$$

Note that we are pretty much using the same idea as we did to solve the examples in the lecture notes, except this time we are figuring out the formula relating the base dimensions from the formula relating the base units. In the examples, we used the formula relating the base dimensions to figure out the formula relating the base units.

(b) Let m be mass and a be acceleration.

Mass has the dimension M. From the table of zero-order units for common dimensions (found in lecture notes), acceleration has zero-order units of $\mathrm{m/s^2}$. Realizing that metres and seconds are base units and using the same idea as in part (a), we can figure out that

$$[a] = \frac{L}{T^2}$$

Then notice that

$$[m][a] = (M)\left(\frac{L}{T^2}\right) = \frac{ML}{T^2} = [F]$$

This relationship suggests that a formula for force is

$$F = ma$$

12. From the table of zero-order units for common dimensions (found in lecture notes), we see that the Joule, J, is the zero-order unit for energy. In the question, we are given that

$$\mathrm{J} = (\mathrm{N})(\mathrm{m})$$

Question 11 shows us how to write a Newton in terms of base units. Therefore, we can write

$$\mathrm{J} = (\mathrm{N})(\mathrm{m}) = \left(\frac{(\mathrm{kg})(\mathrm{m})}{\mathrm{s}^2}\right)(\mathrm{m}) = \frac{(\mathrm{kg})(\mathrm{m}^2)}{\mathrm{s}^2}$$

We can use this relationship to figure out that

$$[E] = \frac{ML^2}{T^2}$$

Now, from the formula for power, we know that

$$[P] = \frac{[E]}{[t]} = \frac{[E]}{T} = \frac{ML^2}{T^3}$$

Therefore, by replacing $[P]$ with W in the formula above and by replacing each base dimension with the corresponding base unit, we can see that

$$\mathrm{W} = \frac{(\mathrm{kg})(\mathrm{m}^2)}{\mathrm{s}^3}$$

And we can be sure that there should not be any dimensionless factor in this expression because of the coherence of the metric system.

13. (a) From the formula given in the question, we can arrive at the following relationship of dimensions (note that m_1, m_2 are masses and that r is a measurement of distance) :

$$[F] = \frac{[G][m_1][m_2]}{[r]^2} = \frac{[G]MM}{L^2} = \frac{[G]M^2}{L^2}$$

But the force of gravity just has the dimension of *force* (it doesn't matter what type of force it is!). So recall from question 11 that the dimension of force $[F]$ can be written in terms of base dimensions as

$$[F] = \frac{ML}{T^2}$$

Therefore, we can equate both expressions for $[F]$ to get the following equation:

$$\frac{[G]M^2}{L^2} = \frac{ML}{T^2}$$

Now we can just solve the equation for $[G]$ by undoing the multiplication and division of the dimensions of L and M:

$$\left(\frac{[G]M^2}{L^2}\right)(L^2) = \left(\frac{ML}{T^2}\right)(L^2)$$

$$[G]M^2 = \frac{ML^3}{T^2}$$

$$\frac{[G]M^2}{M^2} = \frac{ML^3}{(T^2)(M^2)}$$

$$[G] = \frac{L^3}{MT^2}$$

(b) To express the zero-order unit of G in terms of base units, we can simply replace the base dimensions in the formula for $[G]$ with the corresponding base units:

$$\text{zero-order unit of } G = \frac{\mathrm{m}^3}{(\mathrm{kg})(\mathrm{s}^2)}$$