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Grade 6 Math Circles
Binary and Beyond - Solutions

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Exercise 1

A. We make place value charts to represent the numbers, then write out the appropriate sums of powers of 10.

i.

Place Value	10^5	10^4	10^3	10^2	10^1	10^0
Place Value	$100,000$	$10,000$	$1,000$	100	10	1
Digit	1	0	2	0	7	9

$\Rightarrow 102079 = (1)(10^5) + (2)(10^3) + (7)(10^1) + (9)(10^0)$

ii.

Place Value	10^2	10^1	10^0
Place Value	100	10	1
Digit	5	9	3

$\Rightarrow 593 = (5)(10^2) + (9)(10^1) + (3)(10^0)$

iii. $60000 - 5999 = 54001$

Place Value	10^4	10^3	10^2	10^1	10^0
Place Value	$10,000$	$1,000$	100	10	1
Digit	5	4	0	0	1

$\Rightarrow 54001 = (5)(10^4) + (4)(10^3) + (1)(10^0)$

B. Once again, we make place value charts to represent the sums, and then write out the numbers in standard form by reading off the Digits row.

i.

Place Value	10^4	10^3	10^2	10^1	10^0
Place Value	$10,000$	$1,000$	100	10	1
Digit	7	0	4	3	0

$\Rightarrow 70430$

ii.

Place Value	10^5	10^4	10^3	10^2	10^1	10^0
Place Value	$100,000$	$10,000$	$1,000$	100	10	1
Digit	9	0	2	0	8	$1+4=5$

$\Rightarrow 902085$

iii.

$$
\begin{aligned}
(6)(10^3)+(4)(10^2)+(10)(10^1)+(3)(10^0) &= (6)(10^3)+(4)(10^2)+(1)(10^2)+(3)(10^0)\\
&= (6)(10^3)+(5)(10^2)+(3)(10^0)
\end{aligned}
$$

Place Value	10^3	10^2	10^1	10^0
Place Value	$1,000$	100	10	1
Digit	6	5	0	3

$\Rightarrow 6503$

Exercise 2

A. To read the binary numbers out loud, say each digit in the number from left to right.

   i. 101000 $\rightarrow$ one-zero-one-zero-zero-zero

   ii. 100111 $\rightarrow$ one-zero-zero-one-one-one

   iii. 101010 $\rightarrow$ one-zero-one-zero-one-zero

A binary number x is greater than another binary number y if x has a 1 in a column further left than y does. This is due to place value. It is the same idea as, in the decimal system, a 1 in the hundreds column is greater than any combination of digits in just the tens and ones columns. Therefore, we get that

$$100111 < 101000 < 101010$$

B. We make place value charts to represent the binary numbers, and then add up the entries in the Place Value row that have 1s below in the Digits row:

i.

Place Value	2^5	2^4	2^3	2^2	2^1	2^0
Place Value	32	16	8	4	2	1
Digit	1	1	0	0	1	1

Therefore, $110011 \to 32 + 16 + 2 + 1 = 51$

ii.

Place Value	2^3	2^2	2^1	2^0
Place Value	8	4	2	1
Digit	1	1	1	1

Therefore, $1111 \to 8 + 4 + 2 + 1 = 15$

iii. In this case, we only need to compute the place value for the only 1 in the binary number. The place value is $2^9 = 512$. Therefore, $1000000000 \to 512$

Exercise 3

A. Step 1: The greatest power of 2 less than or equal to 45 is $2^5 = 32$.

Step 2: Construct the place value chart with the left-most column being the 2^5 place value:

Place Value	2^5	2^4	2^3	2^2	2^1	2^0
Place Value	32	16	8	4	2	1
Digit						

Step 3: Running total $= 45 - 32 = 13$.

Place Value	2^5	2^4	2^3	2^2	2^1	2^0
Place Value	32	16	8	4	2	1
Digit	1					

Now instead of proceeding to Step 4, notice that we are really just trying to represent 13 in binary. But we have already done that in Example 3! So we can use the answer from Example 3 to fill in the rest of the chart:

Place Value	2^5	2^4	2^3	2^2	2^1	2^0
Place Value	32	16	8	4	2	1
Digit	1		1	1	0	1

Step 5: Fill in the remaining spaces with zeros:

Place Value	2^5	2^4	2^3	2^2	2^1	2^0
Place Value	32	16	8	4	2	1
Digit	1	0	1	1	0	1

Step 6: The binary form of 45 is 101101.

B. Step 1: The greatest power of 2 less than or equal to 128 is $2^7 = 128$. We are pretty much finished already!

Step 2: Construct the place value chart with the left-most column being the 2^7 place value:

Place Value	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
Place Value	128	64	32	16	8	4	2	1
Digit								

Step 3: Running total $= 128 - 128 = 0$.

Place Value	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
Place Value	128	64	32	16	8	4	2	1
Digit	1							

Step 5: Fill in the remaining spaces with zeros:

Place Value	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
Place Value	128	64	32	16	8	4	2	1
Digit	1	0	0	0	0	0	0	0

Step 6: The binary form of 128 is 10000000.

C. Step 1: The greatest power of 2 that is less than or equal to 30 is $2^4 = 16$.

Step 2: Construct a place value chart with the left-most column being the 2^4 place value:

Place Value	2^4	2^3	2^2	2^1	2^0
Place Value	16	8	4	2	1
Digit					

Step 3: Running total $= 30 - 16 = 14$.

Place Value	2^4	2^3	2^2	2^1	2^0
Place Value	16	8	4	2	1
Digit	1				

Step 4: The greatest power of 2 less than or equal to 14 is $2^3 = 8$.

Running total $= 14 - 8 = 6$.

Place Value	2^4	2^3	2^2	2^1	2^0
Place Value	16	8	4	2	1
Digit	1	1			

Step 5: The greatest power of 2 less than or equal to 6 is $2^2 = 4$.
Running total $= 6 - 4 = 2$.

Place Value	2^4	2^3	2^2	2^1	2^0
Place Value	16	8	4	2	1
Digit	1	1	1		

The greatest power of 2 less than or equal to 2 is $2^1 = 2$.
Running total $= 2 - 2 = 0$.
Fill in remaining space with zero.

Place Value	2^4	2^3	2^2	2^1	2^0
Place Value	16	8	4	2	1
Digit	1	1	1	1	0

Step 6: The binary form of 30 is 11110.

Problem Set

1. Notice that we already have a sum of powers of 2, just not quite in the correct form.

$$\begin{aligned}(1)(2^6) + (4)(2^2) + (1)(2^1) &= (1)(2^6) + (2^2)(2^2) + (1)(2^1) \\ &= (1)(2^6) + (1)(2^4) + (1)(2^1)\end{aligned}$$

We can fill in a place value chart as follows:

Place Value	2^6	2^5	2^4	2^3	2^2	2^1	2^0
Place Value	64	32	16	8	4	2	1
Digit	1	0	1	0	0	1	0

By reading off the Digit row, we see that the binary number is 1010010.

2. This question is asking for the number in base 10. All we need to do is simplify the expression:

$$\begin{aligned}(1)(2^4) + (1)(2^2) + (1)(2^0) &= (1)(16) + (1)(4) + (1)(1)\\ &= 16 + 4 + 1\\ &= 21\end{aligned}$$

Therefore, we see that the in the base 10 number system, $(1)(2^4)+(1)(2^2)+(1)(2^0) = 21$.

3. Notice that we already have a sum of powers of 8 and that none of the powers of 8 are multiplied by an integer greater than 7. So we can write $(5)(8^5) + (3)(8^4) + (7)(8^1)$ in a place value chart as follows:

Place Value	8^5	8^4	8^3	8^2	8^1	8^0
Digit	5	3	0	0	7	0

By reading off the Digits row, we see that the base 8 number is 530070.

4. First, we write the base 3 number 210210 in a place value chart:

Place Value	3^5	3^4	3^3	3^2	3^1	3^0
Place Value	243	81	27	9	3	1
Digit	2	1	0	2	1	0

Now we can easily see how to write the base 3 number as a sum of powers of 3:

$$\begin{aligned}210210 &\to (2)(3^5) + (1)(3^4) + (0)(3^3) + (2)(3^2) + (1)(3^1) + (0)(3^0)\\ &= (2)(243) + (1)(81) + (0)(27) + (2)(9) + (1)(3) + (0)(1)\\ &= 486 + 81 + 18 + 3\\ &= 588\end{aligned}$$

So the base 3 number 210210 can be written as the decimal number 588.

5. Using the same method as in Exercise 2, we convert all the binary numbers to get the following sequence of base 10 numbers:

13-1-20-8 9-19 6-21-14

Matching the base 10 numbers to the corresponding letters of the alphabet, we uncover the secret message:

6. There are only two possible 3-digit palindromes in the binary system: 101 and 111
   In the decimal system, we can make 3-digit palindromes as follows:

101	202	303	404	505	606	707	808	909
111	212	313	414	515	616	717	818	919
121	222	323	424	525	626	727	828	929
131	232	333	434	535	636	737	838	939
141	242	343	$\ddots$					$\vdots$
151	252	353		$\ddots$				$\vdots$
161	262	363			$\ddots$			$\vdots$
171	272	373				$\ddots$		979
181	282	383					888	989
191	292	393	$\dots$	$\dots$	$\dots$	797	898	999

   By organizing the palindromes in this way, we see that we can make a 10×9 rectangle of 3-digit palindromes. That is, there are 90 possible 3-digit palindromes in the decimal system.

   Therefore, there are $90 - 2 = 88$ more 3-digit palindromes in the decimal system than the binary system.

7. The Mayans probably learned to count with their fingers and thumbs as well as their toes! There really is no definite answer to this question. It's just good to think about how number systems are created!

   Using the same idea, if we only counted with our fingers (not thumbs), then we would most likely have a base 8 number system.

8. (a) The base 5 number system is a base B number system with $B = 5$. So right from the definition of a base B number system, we know that the place values of the base 5 number system are powers of 5.

   (b) Using the same logic, digits in the base 5 number system may be any of the integers $0, 1, 2, 3, 4$.

   (c) To convert these base 10 numbers to base 5, we need to write them as sums of powers of 5. To do this, we can follow a modified version of the method we used in Example 3 to convert base 10 numbers to binary.

i. The greatest power of 5 less than or equal to 700 is $5^4 = 625$. Therefore, we can write

$$700 = 625 + 75 = (1)(5^4) + 75$$

The greatest power of 5 less than or equal to 75 is $5^2 = 25$. Also, note that $(3)(25) = 75$. Thus, we can write

$$700 = (1)(5^4) + (3)(25) = (1)(5^4) + (3)(5^2)$$

So now we can create a place value chart to represent 700 in base 5:

Place Value	5^4	5^3	5^2	5^1	5^0
Place Value	625	125	25	5	1
Digit	1	0	3	0	0

Thus, by reading off the Digits row from left to right, we see that $700 \to 10300$.

ii. The greatest power of 5 less than or equal to 127 is $5^3 = 125$. Therefore, we can write

$$127 = 125 + 2 = (1)(5^3) + 2$$

The greatest power of 5 less than or equal to 2 is $5^0 = 1$. Clearly, $(2)(1) = 2$. Thus, we can write

$$127 = (1)(5^3) + (2)(1) = (1)(5^3) + (2)(5^0)$$

So now we can create a place value chart to represent 127 in base 5:

Place Value	5^3	5^2	5^1	5^0
Place Value	125	25	5	1
Digit	1	0	0	2

Thus, by reading off the Digits row from left to right, we see that $127 \to 1003$.

iii. The greatest power of 5 less than or equal to 73 is $5^2 = 25$. Also, note that $(2)(25) = 50$. Therefore, we can write

$$73 = 50 + 23 = (2)(25) + 23 = (2)(5^2) + 23$$

The greatest power of 5 less than or equal to 23 is $5^1 = 5$. Also, note that

$(4)(5) = 20$. Therefore, we can write

$$73 = (2)(5^2) + 20 + 3 = (2)(5^2) + (4)(5) + 3 = (2)(5^2) + (4)(5^1) + 3$$

The greatest power of 5 less than or equal to 3 is $5^0 = 1$. Clearly, $(3)(1) = 3$. So we can write

$$73 = (2)(5^2) + (4)(5^1) + (3)(1) = (2)(5^2) + (4)(5^1) + (3)(5^0)$$

So now we can create a place value chart to represent 73 in base 5:

Place Value	5^2	5^1	5^0
Place Value	25	5	1
Digit	2	4	3

Thus, by reading off the Digits row from left to right, we see that $73 \to 243$.

9. Answers may vary.

10. Clearly, the binary numbers 0 and 1 are equal to the base 10 numbers 0 and 1.

    (a) In base 10, $0 + 0 = 0$. Therefore, in binary, $0 + 0 = 0$.

    (b) In base 10, $0 + 1 = 1$. Therefore, in binary, $0 + 1 = 1$.

    (c) In base 10, $1 + 0 = 1$. Therefore, in binary, $1 + 0 = 1$.

    (d) In base 10, $1 + 1 = 2$. The decimal number 2 is the same as the binary number 10. Therefore, in binary, $1 + 1 = 10$.

    (e) In base 10, $1+1+1 = 3$. The decimal number 3 is the same as the binary number 11. Therefore, in binary, $1 + 1 + 1 = 11$.

11. When doing vertical addition of binary numbers, if there is a sum of $1 + 1 = 10$ in a column, carry the 1 to the next column to the left (just like carrying in decimal addition). Using this idea and the results from the previous question, we get the following:

    (a) $\begin{array}{r} 10 \\ +1 \\ \hline 11 \end{array}$

(b) $$\begin{array}{r} 1001 \\ +110 \\ \hline 1111 \end{array}$$

(c) $$\begin{array}{r} \color{red}{1} \\ 10 \\ +10 \\ \hline 100 \end{array}$$

(d) $$\begin{array}{r} \color{red}{11} \\ 110 \\ +10 \\ \hline 1000 \end{array}$$

(e) $$\begin{array}{r} \color{red}{111} \\ 111 \\ +11 \\ \hline 1010 \end{array}$$

Note: Red numbers are digits that were carried.

12. (a) There are a few ways to think about this problem. The easiest is to recognize that the largest 5-digit binary number is 11111, and the largest 4-digit binary number is 1111. So there are $11111 - 1111$ different 5-digit binary numbers.

    But $11111 - 1111$ is the same as $(11111 + 1) - (1111 + 1) = 100000 - 10000$. Or in base 10, the difference is $2^5 - 2^4 = 32 - 16 = 16$. That is, there are 16 different 5-digit binary numbers.

    (b) It is helpful to realize that all binary numbers whose last digit is a 1 are odd, and all binary numbers whose last digit is a 0 are even. Since there is an even number (16) of different 5-digit binary numbers, half of them must be odd. Therefore, there are 8 different 5-digit binary numbers that have 1 as their last digit.

    (c) We can use the same idea that we used in part (a) to solve this. The only difference is that the largest 5-digit base B number possible is the 5-digit number whose digits are all $(B - 1)$. This number is still 1 less than the number 100000. So there are $100000 - 10000$ different 5-digit base B numbers. Or in base 10, we can evaluate this as $B^5 - B^4$ different 5-digit base B numbers.

13. (a) With a little bit of thought, we can just modify our solutions from the previous question.

The largest n-digit binary number is the n-digit binary number whose digits are all ones.

$$\underbrace{111\cdots 11}_{n \text{ ones}}$$

Similarly, the largest $(n-1)$-digit binary number is the $(n-1)$-digit binary number whose digits are all ones.

$$\underbrace{111\cdots 11}_{(n-1) \text{ ones}}$$

Therefore, subtracting the smaller number from the larger number will give the amount of different n-digit binary numbers. But like before, we can do less work if we simplify the subtraction by adding 1 to each number. That is, the amount of different n-digit binary numbers is given by the following difference:

$$1\underbrace{00\cdots 00}_{n \text{ zeros}} - 1\underbrace{00\cdots 00}_{(n-1) \text{ zeros}}$$

So just like before, we convert to base 10 to evaluate the difference as $2^n - 2^{n-1}$. That is, there are $2^n - 2^{n-1}$ different n-digit binary numbers.

You can check this expression by substituting in $n = 5$ and comparing the result to your answer from 12.(a).

(b) Just as in the 5-digit case, we find that there is an even number of different n-digit binary numbers. This is because $2^n - 2^{n-1}$ is an even number for all $n \geq 2$ since it is a difference of even numbers (all powers of 2 with an exponent greater than or equal to 1 are even numbers).

Since there is an even number of different n-digit binary numbers, half of them must be odd (have 1 as last digit). Therefore, there are $\left(\frac{1}{2}\right)(2^n - 2^{n-1})$ different n-digit binary numbers that have 1 as their last digit.

You can also prove this fact to yourself by writing out consecutive binary numbers and noticing the alternating pattern of the last digit.

(c) This question requires you to put the ideas from 12.(c) and 13. together.

With a little bit of clever thinking, we realize that the amount of different n-digit base B numbers is always given by the expression below:

$$1\underbrace{00\cdots00}_{n\text{ zeros}} - 1\underbrace{00\cdots00}_{(n-1)\text{ zeros}}$$

However, these numbers are not always binary numbers. These are numbers in their base B form. So this expression has different values depending on the base B.

By converting the expression to base 10, we get that there are $B^n - B^{n-1}$ different n-digit base B numbers.

14. The important numbers in the binary system are powers of 2, the place values.

    The key idea to this solution is to pair each stack of coins with a certain binary place value. This is done by lining up the stacks so that from left to right we can call them stacks 0 through 6, and then taking $2^0 = 1$ coin from stack 0, $2^1 = 2$ coins from stack 1, $2^2 = 4$ coins from stack 2, and so on. Note that there will be enough coins in each stack for this strategy to work, since the most coins we take is $2^6 = 64$, which is less than 100.

    Next, take the selected coins from each stack and place them in piles on the balance all at once. Record the mass. The mass measured will depend on which stacks were counterfeit. Let m represent the measured mass.

    If all the coins were real, we would have measured $(1+2+4+8+16+32+64)(10) = (127)(10) = 1270$ grams. Because there will be some counterfeit coins on the balance, the actual mass measured, m, will be greater than this. If we subtract 1270 from the m, then we can think of the counterfeit coins as weighing 1 gram and the real coins as weighing nothing. This makes things simpler.

    The difference $m - 1270$ will be a number that is the sum of two of the seven possible powers of 2. Because it is a sum of powers of 2, we can easily write it as a binary number. Also, the columns with a 1 in the binary number correspond to the counterfeit stacks!

    For example, if $m - 1270 = 40$, then we would write 40 as a binary number. We would get that $40 = 32 + 8 = 2^5 + 2^3 \to 101000$. There is a 1 in the 2^3 and 2^5 place values, so stacks 3 and 5 are counterfeit.

    Note that there will only ever be one way to write $m - 1270$ as a binary number. This follows from the properties of binary numbers. Therefore, we will always be able to determine which two stacks are counterfeit for any possible value of $m - 1270$!