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Grade 6 Math Circles

November 12/13, 2013

Divisibility - Solutions

Example

~~1~~	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~

The prime numbers less than 100 are: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89 and 97.

Problem Set Problem Set

1.

	2	3	4	5	6	7	8	9	10	11	12
272	☒	☐	☒	☐	☐	☐	☒	☐	☐	☐	☐
396	☒	☒	☒	☐	☒	☐	☐	☒	☐	☒	☒
926	☒	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
5 940	☒	☒	☒	☒	☒	☐	☐	☒	☒	☒	☒
6 048	☒	☒	☒	☐	☒	☒	☒	☒	☐	☐	☒
7 848	☒	☒	☒	☐	☒	☐	☒	☒	☐	☐	☒
22 176	☒	☒	☒	☐	☒	☒	☒	☒	☐	☒	☒
39 208	☒	☐	☒	☐	☐	☐	☒	☐	☐	☐	☐
479 001 600	☒	☒	☒	☒	☒	☒	☒	☒	☒	☒	☒

2. The number 6 has four divisors: 1, 2, 3, and 6.
   The number 8 has four divisors: 1, 2, 4, and 8.
   The number 9 has three divisors: 1, 3, and 9.
   The number 10 has four divisors: 1, 2, 5, and 10.
   The number 12 has six divisors: 1, 2, 3, 4, 6, and 12.

   Therefore the smallest positive number with 6 divisors is 12.

3. After finding all the factors of 36, the possibilities are:

   - 1 group of 36 students,
   - 2 groups of 18 students,
   - 3 groups of 12 students,
   - 4 groups of 9 students,
   - 6 groups of 6 students,
   - 9 groups of 4 students,
   - 12 groups of 3 students,
   - 18 groups of 2 students, and
   - 36 groups of 1 student.

4. First, find the prime factorization of 1925: $5 \times 5 \times 7 \times 11$

   Now combine, with multiplication only, the factors into 3 numbers in every possible way so that the product of those three numbers is still 1925. Remember, having sides of length 1 are acceptable.

The different possible dimensions are:

- 1cm × 1cm × 1925cm
- 1cm × 35cm × 55cm
- 1cm × 5cm × 385cm
- 1cm × 7cm × 275cm
- 1cm × 11cm × 175cm
- 1cm × 25cm × 77cm
- 5cm × 5cm × 55cm
- 5cm × 7cm × 55cm
- 5cm × 11cm × 35cm
- 7cm × 11cm × 25cm

5. A number is divisible by 11 if "the sum of every second digit less the remaining digits is divisible by 11".

   Thus $2 + 4 + 6 + 8 - 1 - 3 - \square - 7$ must be divisible by 11.

   Simplifying, we get that $9 - \square$ must be divisible by 11.

   What positive, single-digit whole number can be subtracted from 9 to get a number that is divisible by 9?

   The positive multiples of 11 are 0, 11, 22, 33, ...

   Thus we can only subtract 9 from 9 to get 0, which is divisible by 11.

   Therefore the digit $\square$ must be 9.

6. A number is divisible by 4 if "the last two digits are divisible by 4".

   The number formed by the last two digits is $z2$. For what value of z is $z2$ divisible by 4? 12, 32, 52, 72, ... are all divisible by 4. For this solution, let's take $z = 1$.

   Our number is now $29x54y21412$.

   A number is divisible by 9 if "the sum of the digits is divisible by 9".

   The sum of our digits is $2 + 9 + x + 5 + 4 + y + 2 + 1 + 4 + 1 + 2 = 30 + x + y$. What two digits can be added to 30 such that the sum is divisible by 9?

   The smallest number greater than 30 that is divisible by 9 is 36 ($36 \div 9 = 4$). Thus finding values for x and y such that $x + y = 6$ will make $29x54y21412$ divisible by 9.

We can take $x = 0, y = 6$, or $x = 1, y = 5$, or $x = 2, y = 4$, ... For this solution, let's take $x = 3, y = 3$.

Thus our number is 29354321412.

There are many more solutions.

7. (a) A number is divisible by 2 if "the last digit of the number is even".

   5 is not even, but 6 is. Thus you must add 1 to 375 to make it divisible by 4.

   (b) A number is divisible by 3 if "the sum of all the digits is divisible by 3".

   The sum of the digits in 375 is $3 + 7 + 5 = 15$. 15 is already divisible by 3, so nothing has to be added to the number.

   (c) A number is divisible by 4 if "the last two digits are divisible by 4".

   The number formed by the last two digits is 75. 75 is not divisible by 4, but 76 is ($76 \div 4 = 19$). Thus you must add 1 to 375 to make it divisible by 4.

   (d) A number is divisible by 5 if "the number ends in 5 or 0".

   375 already ends with a 5, so nothing has to be added to the number.

   (e) A number is divisible by 6 if "the number is divisible by 2 *and* 3".

   As we already figured out, 375 is not divisible by 2, but it is divisible by 3; so 375 is not divisible by 6.

   We must add some multiple of 3 (so that the number will remain divisible by 3) until we get an even last digit.

   Adding 3 once will give us 378. 8 is even, so 378 is divisible by 2. $3 + 7 + 8 = 18$ is also divisible by 3.

   Thus you must add 3 to 375 to make it divisible by 6.

   (f) A number is divisible by 7 if "twice the last digit subtracted from the remaining digits is divisible by 7".

   $37 - 2 \times 5 = 27$ is not divisible by 7, so 375 is also not divisible by 7.

   Two times what number greater than 5 *is* divisible by 7?

   Try 6: $37 - 2 \times 6 = 25$ is not divisible by 7, so 376 is also not divisible by 7.

   Try 7: $37 - 2 \times 7 = 23$ is not divisible by 7, so 377 is also not divisible by 7.

   Try 8: $37 - 2 \times 8 = 21$ IS divisible by 7, so 378 is also divisible by 7.

   Thus you must add 3 to 375 to make it divisible by 7.

8. * The desired number is the product of 12 and some other factor.

Since $12 = 2^2 \times 3$, the number is divisible by 2^2 and by 3.

The smallest number with only 1's and 0's that is divisible by 2 is $10 = 2 \times 5$.

Since there are two 2's, there must also be two 5's if we are to have only 1's and 0's, so the number contains 5^2.

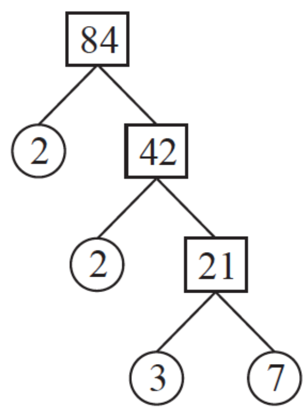
Also, for a number to be divisible by 3, the sum of its digits must be divisible by 3, and since we are allowed only 1's and 0's there must be three 1's.

If the number contains 2^2 and 5^2 it must end in 00 since $2^2 \times 5^2 = 100$.

If it has three 1's, the smallest possible value is 11 100.

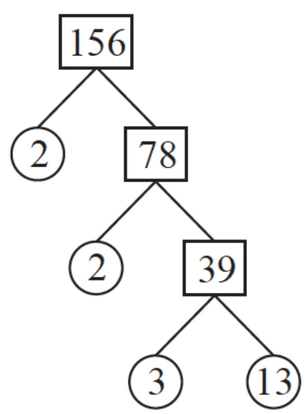
By checking, $11\ 100 = 12 \times 925$, so it satisfies the requirements.

9. (a)



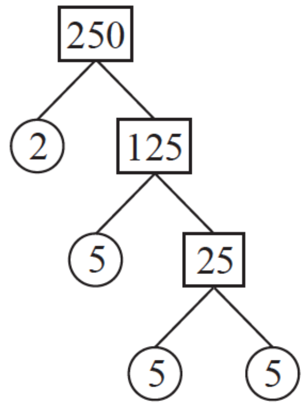
Thus $84 = 2 \times 2 \times 3 \times 7$.

(b)



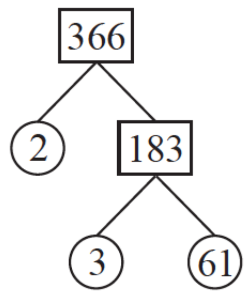
Thus $156 = 2 \times 2 \times 3 \times 13$.

(c)



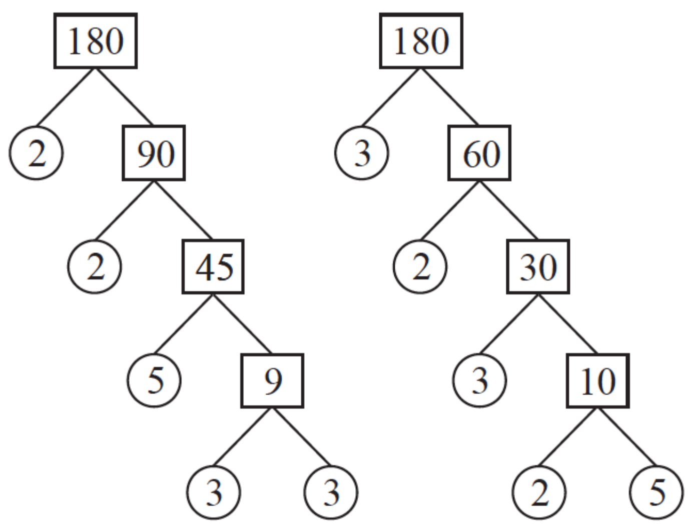
Thus $250 = 2 \times 5 \times 5 \times 5$.

(d)



Thus $366 = 2 \times 3 \times 61$.

10.



Thus $180 = 2 \times 2 \times 5 \times 3 \times 3$ and $180 = 3 \times 2 \times 3 \times 2 \times 5$.

Notice that the prime factorizations are identical (because multiplication is *commutative*, which means that order doesn't matter), even though the trees are different.

11. There are three ways of multiplying two numbers together to get 81:

- 1×81
- 3×27

- 9×9

But $1 + 81 \neq 30$, $9 + 9 \neq 30$, and $3 + 27 = 30$. So the two numbers of interest are 3 and 27.

12. There are nine ways of multiplying two numbers together to get 300:

    - $1 \times 300 \quad \Rightarrow \quad 1 + 300 = 301$
    - $2 \times 150 \quad \Rightarrow \quad 2 + 150 = 152$
    - $3 \times 100 \quad \Rightarrow \quad 3 + 100 = 103$
    - $4 \times 75 \quad \Rightarrow \quad 4 + 75 = 79$
    - $5 \times 60 \quad \Rightarrow \quad 5 + 60 = 65$
    - $6 \times 50 \quad \Rightarrow \quad 6 + 50 = 56$
    - $10 \times 30 \quad \Rightarrow \quad 10 + 30 = 40$
    - $12 \times 25 \quad \Rightarrow \quad 12 + 25 = 37$
    - $15 \times 20 \quad \Rightarrow \quad 15 + 20 = 35$

13. * Because 12 is a multiple of 4 ($4 \times 3 = 12$) AND 6 ($6 \times 2 = 12$). the signs will flash at the same time every 12 seconds.

    There are 60 seconds in a minute, so the signs will flash at the same time $60 \div 12 = 5$ times.

14. ** The prime factors of 36 are 2, 2, 3, 3; in other words, $2 \times 2 \times 3 \times 3 = 36$.

    By finding the prime factorization of 36, we can construct the following table more easily. The following are all possible ages for the three siblings:

Age 1	Age 2	Age 3	Sum
1	1	36	38
1	2	18	21
1	3	12	16
1	4	9	14
1	6	6	13
2	2	9	13
2	3	6	11
3	3	4	10

Because we are not told the day of the month on which my friend's birthday falls, we know that knowing the sum of the ages does not give a definitive answer; thus, there must be more than one solution with the same total.

Only two sets of possible ages add up to the same totals:

- $1 \times 6 \times 6 = 36$
- $2 \times 2 \times 9 = 36$

In the first case, there is no 'eldest sibling' - two siblings are aged six. Therefore, when told that one sibling is the eldest, my friend should be able to conclude that the correct solution is the second case.

15. ** Two numbers are *co-prime* if they do not have any prime factors in common. List all possible numbers that are co-prime with the following numbers that are < 50.

    (a) The prime factorization of 390 is $390 = 2 \times 3 \times 5 \times 13$.

    The following are all the numbers less than 50 that do not share any of the factors with 390:

    7, 11, 17, 19, 23, 29, 31, 37, 41, 43, 47, 49

    (b) The prime factorization of 210 is $210 = 2 \times 3 \times 5 \times 7$.

    The following are all the numbers less than 50 that do not share any of the factors with 210:

    11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47

    (c) The prime factorization of 49335 is $49335 = 3 \times 5 \times 11 \times 13 \times 23$.

    The following are all the numbers less than 50 that do not share any of the factors with 49335:

    2, 4, 7, 8, 14, 16, 17, 19, 23, 29, 31, 32, 34, 37, 38, 41, 43, 47, 49

16. Determine the greatest common factors of the following numbers.

    (a) Prime factorization method:

    The prime factorization of 231 is $231 = 3 \times 7 \times 11$.

    The prime factorization of 660 is $660 = 2 \times 2 \times 3 \times 5 \times 11$.

    To find the greatest common factor of these numbers, we look at all of the prime factors they have in common. Both numbers have one 3 and one 11 as prime factors. Thus their GCF is $3 \times 11 = 33$.

    Euclidean algorithm:

Begin by using long division to divide the bigger number by the smaller one.

$$\begin{array}{r} 2 \\ 231\overline{)660} \\ \underline{462} \\ 198 \end{array}$$

Now take the divisor (231) and divide it by the remainder (198).

$$\begin{array}{r} 1 \\ 198\overline{)231} \\ \underline{198} \\ 33 \end{array}$$

Again, take the divisor (198) and divide it by the remainder (33).

$$\begin{array}{r} 5 \\ 45\overline{)225} \\ \underline{225} \\ 0 \end{array}$$

Stop when the remainder is 0. The last non-zero remainder is the greatest common divisor!

The last non-zero remainder in this question is 33, thus the GCF of 231 and 660 is 33.

(b) 1386 and 322

Prime factorization method:

The prime factorization of 1386 is $1386 = 2 \times 3 \times 3 \times 7 \times 11$.
The prime factorization of 322 is $322 = 2 \times 7 \times 23$.

To find the greatest common factor of these numbers, we look at all of the prime factors they have in common. Both numbers have one 2 and one 7 as prime factors. Thus their GCF is $2 \times 7 = 14$.

Euclidean algorithm:

Begin by using long division to divide the bigger number by the smaller one.

$$\begin{array}{r} 4 \\ 322\overline{)1386} \\ \underline{1288} \\ 98 \end{array}$$

Now take the divisor (322) and divide it by the remainder (98).

$$\begin{array}{r}
3 \\
98\,\overline{)\,322} \\
\underline{294} \\
28
\end{array}$$

Again, take the divisor (98) and divide it by the remainder (28).

$$\begin{array}{r}
3 \\
28\,\overline{)\,98} \\
\underline{84} \\
14
\end{array}$$

Again, take the divisor (28) and divide it by the remainder (14).

$$\begin{array}{r}
2 \\
14\,\overline{)\,28} \\
\underline{28} \\
0
\end{array}$$

Stop when the remainder is 0. The last non-zero remainder is the greatest common divisor!

The last non-zero remainder in this question is 14, thus the GCF of 1386 and 322 is 14.

17. The GCF of two numbers a and b equal to a or b if a is divisible by b or b is divisible by a.

18. * First, we must find the greatest common factor of 72, 84, and 48.

    The hint gave us the fact that $GCF(a, b, c) = GCF(a, GCF(b, c))$. In this case, it means that $GCF(72, 84, 48) = GCF(72, GCF(84, 48))$.

    Let us first find the GCF of 84 and 48.

    Using prime factorization, we get that $84 = 2 \times 2 \times 3 \times 7$ and $48 = 2 \times 2 \times 2 \times 2 \times 3$.

    To find the greatest common factor of these numbers, we look at all of the prime factors they have in common. Both numbers have two 2s and one 3 as prime factors. Thus their GCF is $2 \times 2 \times 3 = 12$.

    Now find the GCF of 72 and 12.

    Using the Euclidean algorithm,

$$\begin{array}{r}
6 \\
12\,\overline{)\,72} \\
\underline{72} \\
0
\end{array}$$

The remainder is already 0, so the greatest common factor of 72 and 12 is 12, and thus the greatest common factor of 72, 84, and 48 is 12.

This means that Sally can create at most 12 loot bags.

Divide the number of candy bars, lollipops, and gum balls by 12 to find out how many of each will be in every loot bag.

$$72 \div 12 = 6$$

$$84 \div 12 = 7$$

$$48 \div 12 = 4$$

So, Sally can create 12 loot bags with 6 candy bars, 7 lollipops, and 4 gum balls in each.