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Math Circles: Graph Theory II

Centre for Education in
Mathematics and Computing
March 6, 2013

1. If a connected graph has an Eulerian walk, then every vertex except possibly the endpoints is traversed an even number of times, since one must enter and exit each vertex. Hence every vertex except possibly the two endpoints has even degree. The degrees of the endpoints must be both odd or both even, by **??**.

   Conversely, if a connected graph has 0 vertices of odd degree, then it has an Eulerian circuit by **??**. If a connected graph has 2 vertices of odd degree, then we can join those 2 vertices with a new edge, obtain an Eulerian circuit for the new graph by **??**, and delete the new edge from the Eulerian circuit to obtain an Eulerian walk for the original graph.

2. If we add all the degrees of all the vertices, then we must obtain twice the number of edges, since each edge joins two vertices. Hence the sum of all degrees of all vertices is even. The result follows.

3. If a connected graph has an Eulerian circuit, then every vertex, including the endpoints, is traversed an even number of times, since one must enter and exit each vertex. Hence every vertex has even degree.

   Conversely, if every vertex has even degree, then we can simply take any circuit in the graph that does not repeat any edges. If this circuit includes all the edges, then we are done; otherwise, start (and end) the circuit from a vertex that possesses an edge not in the circuit, and then continue along another circuit starting from the edge not in the circuit. Repeat as necessary until each edge is included once.

4. The algorithm in the second part of **??**, known as *Fleury's algorithm*, suffices to find an Eulerian circuit.

5. Use the algorithm given in the second part of **??**.

6. We use induction on the number of edges in the graph.

   Base case: Suppose the graph has zero edges. Then the number of vertices, edges, faces, and components of the graph is as follows:

vertices	V
edges	0
faces	1
components	V

From this, we find that $V - E + F = V + 1$ and $C + 1 = V + 1$, and the two are equal. This establishes the base case.

Inductive case: Let $E > 0$. Assume that Euler's formula holds for all graphs with fewer than E edges.

Since $E > 0$, the graph has an edge e. Delete e from the graph to obtain a new graph. There are two cases.

(a) Suppose that both sides of e are part of the same face. In this case, the new graph has the following number of vertices, edges, faces, and components:

	original graph	new graph
vertices	V	V
edges	E	$E - 1$
faces	F	F
components	C	$C + 1$

Justification of the above: Clearly the number of vertices of the new graph is V, and the number of edges is $E - 1$. The number of faces is still F, since the two sides of e comprise the same face, so removing e does not alter the number of faces. The number of components increases by 1 upon removal of e, since e connects two otherwise disconnected components.

Observe that the new graph indeed has fewer edges than the original graph. Therefore the induction hypothesis applies, and we may assume that Euler's formula holds for the new graph. In this case, the formula states

$$V - (E - 1) + F = (C + 1) + 1.$$

Rearranging terms, we obtain $V - E + F = C + 1$, as desired.

(b) Now suppose the two sides of e consist of different faces. Then the new graph obtained upon deleting e has the following numbers of vertices, edges, faces, and components:

	original graph	new graph
vertices	V	V
edges	E	$E - 1$
faces	F	$F - 1$
components	C	C

The only difference is that, since e divides two faces, the two sides of e merge into one face after e is removed. This explains why the new graph has $F - 1$ faces. The number of components of the new graph is still C, since e connects two vertices which are still connected without e.

Again, the new graph has fewer than E edges, so the induction hypothesis states that Euler's formula holds for the new graph, or in other words

$$V - (E - 1) + (F - 1) = C + 1,$$

which again simplifies to $V - E + F = C + 1$.

7. A connected graph drawn on the surface of a torus can attain $V - E + F = 0$.