Intermediate Math Circles
March 6, 2013
Number Theory I

What is Number Theory?

A branch of mathematics where mathematicians examine and study patterns found within the *natural number* set (positive integers).
What special number sets/patterns are you already familiar with?
Even Numbers $2, 4, 6, 8, 10, 12, \cdots$
Odd Numbers $1, 3, 5, 7, 9, 11, \cdots$
Square $1, 4, 9, 16, 25, 36, \cdots$
Cube $1, 8, 27, 64, 125, \cdots$
Prime $2, 3, 5, 7, 11, 13, \cdots$
Composite $4, 6, 8, 9, 10, 12, 14, \cdots$

Prime Numbers

A prime number is a positive integer that has **exactly two** positive divisors, 1 and itself.

The most systematic way of identifying and listing the set of prime numbers is to examine the list of natural numbers and remove the composite numbers. This method is called the *Sieve of Erastothenes*.

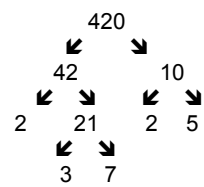
1. The integer 1 is not prime since it has only one positive divisor (i.e. 1), therefore the number 1 does not match our definition of prime.
2. Circle the number 2. This is our first prime number.
3. Move through the list and cross out all multiples of 2.
4. Return back to the start of our list, circle the first number in the list that is not crossed out (i.e. 3).
5. Move through the list and cross out all multiples of 3.
6. Repeat, until we circle the prime number greater than...?
7. Circle all numbers that are not crossed out. These are all Prime!

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120
121	122	123	124	125	126	127	128	129	130
131	132	133	134	135	136	137	138	139	140
141	142	143	144	145	146	147	148	149	150

Fundamental Theorem of Arithmetic

Every integer greater than 1 is either prime or can be written as a unique product of prime numbers. This unique product of prime factors is known as *prime factorization*. Often we use a factor tree to help determine the prime factorization of a number.

For example,
the prime factorization of 420 is $2^2 \times 3 \times 5 \times 7$.



Exercise 1.
Determine the prime factorization of 147.

How Many Primes Are There?

Is there a largest prime number? Does the fact that every integer larger than 1 is either prime or can be written as a product of primes, indicate that there are infinitely many primes? Unfortunately, it doesn't. Perhaps all very large numbers are just written as a product of many primes. Notice that our table of primes reveals that the prime numbers become less dense as we move further through the table.

The largest discovered prime number is $2^{57885161} - 1$. It was found on January 25, 2013. Discovered by Curtis Cooper, a mathematician at the University of Central Missouri. It took him 39 days of non-stop computing to check if the number was indeed prime. This number is the 48th Mersenne Prime. Prime numbers written as $2^n - 1$ are known as Mersenne Primes.

It has been proven that there are **infinitely many** primes. This proof was included in Euclid's Elements.

Prime Factorization of Numbers 1-150.

1	2	3	2^2	5	2×3	7	2^3	3^2	2×5
11	$2^2\times3$	13	2×7	3×5	2^4	17	2×3^2	19	$2^2\times5$
3×7	2×11	23	$2^3\times3$	5^2	2×13	3^3	$2^2\times7$	29	$2\times3\times5$
31	2^5	3×11	2×17	5×7	$2^2\times3^2$	37	2×19	3×13	$2^3\times5$
41	$2\times3\times7$	43	$2^2\times11$	$3^2\times5$	2×23	47	$2^4\times3$	7^2	2×5^2
3×17	$2^2\times13$	53	2×27	5×11	$2^3\times7$	3×19	2×29	59	$2^2\times3\times5$
61	2×31	$3^2\times7$	2^6	5×13	$2\times3\times11$	67	$2^2\times17$	3×23	$2\times5\times7$
71	$2^3\times3^2$	73	2×37	3×5^2	$2^2\times19$	7×11	$2\times3\times13$	79	$2^4\times5$
3^4	2×41	83	$2^2\times3\times7$	5×17	2×43	3×29	$2^3\times11$	89	$2\times3^2\times5$
7×13	$2^2\times23$	3×31	2×47	5×19	$2^2\times3$	97	2×7^2	$3^2\times11$	$2^2\times5^2$
101	$2\times3\times17$	103	$2^3\times13$	$3\times5\times7$	2×53	107	$2^2\times3^3$	109	$2\times5\times11$
3×37	$2^4\times7$	113	$2\times3\times19$	5×23	$2^2\times29$	$3^2\times13$	2×59	7×17	$2^3\times3\times5$
11^2	2×61	3×41	$2^2\times31$	5^3	2×63	127	2^7	3×43	$2\times5\times13$
131	$2^2\times3\times11$	7×19	2×67	$3^3\times5$	$2^3\times17$	137	$2\times3\times23$	139	$2^2\times5\times7$
3×47	2×71	11×13	$2^4 3^2$	5×29	2×73	3×7^2	$2^2\times37$	149	$2\times3\times5^2$

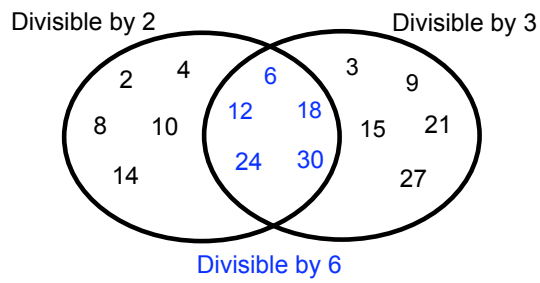
Divisibility Rules

Divisible By	Divisibility Test
2	The number is even. i.e. the ones digit is divisible by 2.
3	The sum of all digits is divisible by 3.
4	The last two digits of the number must be divisible by 4.
5	The ones digit of the number is 0 or 5.
6	Test whether the number is divisible by two and three.
7	Remove the ones digit. From the remaining digits subtract twice the value of the ones digit. If this number is divisible by 7, the original number is also divisible by 7. This process can be repeated until the value reached is easily identified as a multiple of 7 or not.
8	The last three digits of the number must be divisible by 8.
9	The sum of all digits is divisible by 9.
10	The ones digit of the number is 0.
11	Starting with a + sign, place alternating + and - signs on each digit. If the alternating sum is 0 or divisible by 11, then the original number is also divisible by 11.
12	Test whether the number is divisible by three and four.
13	From the remaining digits add four times the value of the ones digit. If this number is divisible by 13, then the original number is also divisible by 13. This process can be repeated until the value reached is easily identified as a multiple of 13 or not.

Exercise 2.
Determine the prime factorization of 882, 1575 and 22022.

Exercise 3.
Without using a calculator, determine the value of k that will cause the 4-digit number $4k78$ to be divisible by 11.

Exercise 4.
Determine the number of positive integers less than 500 that are **not divisible** by 2 or 3.



How Many Divisors Does The Number 100 Have?

The prime factorization of 100 is $2^2 \times 5^2$. The following table will helps us identify all divisors of 100. Complete this table by multiplying each value in a row by each value in a column.

	2^0	2^1	2^2
5^0			
5^1			
5^2			

The Sum of Divisors

The sum of divisors can be found by adding together all of the column sums **or** all of the row sums.

	2^0	2^1	2^2	Row Sum
5^0	1	2	4	$7 = 7 \times 5^0$
5^1	5	10	20	$35 = 7 \times 5^1$
5^2	25	50	100	$175 = 7 \times 5^2$
Column Sum	$31 = 31 \times 2^0$	$62 = 31 \times 2^1$	31×2^2	

$$\begin{aligned}
\text{Sum of Divisors} &= \text{Sum of Row Sums} & &= \text{Sum of Column Sums} \\
&= (7 + 7 \times 5 + 7 \times 25) & &= (31 + 31 \times 2 + 31 \times 4) \\
&= 7 \times 31 & &= 31 \times 7 \\
&= 217 & &= 217
\end{aligned}$$

A Faster Approach to Finding the Sum of Divisors

$$\begin{aligned}
\text{Sum of Divisors} &= 7 \times 31 \\
&= \text{Sum of First Row} \times \text{Sum of First Column}
\end{aligned}$$

Exercise 5.

Determine the sum of divisors of 441 and 4563.

Proper Divisors

A *proper divisor* is a positive integer value that can divide equally into a number but is not the number itself. For example, the proper divisors of 100 are $1, 2, 4, 5, 10, 20, 25, 50$

Exercise 6.

Determine the sum of all proper divisors of 2744.

Abundant, Deficient and Perfect Numbers

An *abundant number* is a positive integer whose sum of proper divisors is greater than the number itself. For example, 12 is an abundant number since the sum of its proper divisors is $1 + 2 + 3 + 4 + 6 = 16 > 12$.

A *deficient number* is a positive integer whose sum of proper divisors is less than the number itself. For example, 15 is a deficient number since the sum of its proper divisors is $1 + 3 + 5 = 9 < 15$.

A *perfect number* is a positive integer whose sum of proper divisors is equal to the number itself. For example, 6 is a perfect number since the sum of its proper divisors is $1 + 2 + 3 = 6$.

Exercise 7.
Identify the following numbers as abundant, deficient or perfect.
(a) 28 (b) 32 (c) 78

Euclid's Method For Finding Perfect Numbers

Let p be a prime number.
If p is prime, we then check to see if $2^p - 1$ is a Mersenne Prime.
If $2^p - 1$ is a Mersenne Prime, then a perfect number can be found using the formula $2^{p-1}(2^p - 1)$.

For example, when $p = 2$, $2^p - 1 = 2^2 - 1 = 3$ A Mersenne Prime!
The perfect number found is $2^{2-1}(2^2 - 1) = 2^1(3) = 6$.

Exercise 8. Find the next perfect number.