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Grade 11/12 Math Circles
Symbolic Logic I
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INTRODUCTION

Logic and common sense are not necessarily the same thing. If you were in a forest and saw some bear tracks on the ground, you might guess there is a bear around, and go somewhere else to stay safe. A logician, following pure logic, might not! They might say "Hmm... I see what appears to be bear tracks, but I'm not sure what caused them. It might be a bear, I'll give you that, but it might also be an odd wind pattern, or maybe there was a bear here a long time ago, and the tracks remained.. or maybe there were some kids trying to play a joke, with fake bear tracks. I am not certain that these tracks imply there is a bear around, so there is no reason to believe there is one." Of course, this sounds a little naive, since, as humans, we have evolved to play it safe in such situations, but this is the reason why some people don't always follow proper logical conclusions. We are programmed to have slightly fuzzy logic, so that we often assume that what is likely the case is probably indeed the case. So, before we begin, please do understand that the world of logic is very black-and-white. There is no grey area. A statement can either be true or false, and cannot be both at the same time. There's no real "maybe" in logic. We deal with absolute, pure truths. With this in mind, we take statements for what they are (and no more than what they are).

Suppose (in this hypothetical, logically-sound world) that the Math Circles are 24 hours long, they take place every Wednesday in the MC building, and nobody ever gets sick or misses one, and no one ever even leaves the building during those 24 hours (hypothetical) that are Wednesday.

Now, suppose you fall asleep one day and wake up all groggy and you don't know where you are. You don't recognize your surroundings (because you are all groggy) but you look at a watch that tells you the day, and it's Wednesday. Then you know for certain (following the logic of the previous paragraph) that you are in the MC building, because you HAVE to be there if it's a Wednesday.

On the other hand, suppose you fall asleep and wake up all groggy, and you don't know which day it is. But you DO know that you are not in MC. Is it Wednesday? NO! It cannot be, because if it were Wednesday, you would have to be in the MC building. Since you're not in the MC building, it cannot be Wednesday.

Lastly, suppose you wake up all groggy and you don't know which day it is, but you find yourself in the MC building. What day is it? You might think "Wednesday", because there is definitely some relationship with being in MC and it being Wednesday. But here is where you need to be careful. Nobody said anything about where you are on the days that aren't Wednesday. Nowhere did we say you are somehow forbidden from MC on days that are not Wednesdays. Maybe you have some

"Physics Circles" on Mondays, and they also take place in MC. You can't conclude anything, except that it is *possible* it's Wednesday, but not for sure.

The above three scenarios point out that when you have an "if/then" statement, such as "If A then B", you can follow the logic that if you do indeed have A, then B must follow. And similarly, if you know that B is not the case, then A cannot possibly be the case either. But you need to be careful, and remember that if you have B, then you cannot necessarily conclude that A is true. It might, but it might not be. We'll explore this a little more later.

CONNECTIVES

We will take letters such as P and Q to represent statements with well-defined "truth values" (that is, P can be true, or P can be false, but not both). There are some symbols which represent some basic logical connectives. A "connective" is simply something which joins two statements, or modifies a single statement, with a known effect on the truth values. We will state what they are, how to denote them symbolically, and exactly what they mean as far as logic goes.

Not

The simplest connective is the negation or "not" connective. It is denoted with the symbol $\sim$, so that $\sim P$ means "not P". It is defined (as naturally expected) to be false if P is true, and true if P is false. We summarize this in a "truth table":

P	$\sim P$
T	F
F	T

Example: If P is the statement "I am hungry", then $\sim P$ is the statement "I am not hungry" (or, if you want to get really technical, "It is not the case that I am hungry").

And

The next simplest connective is the "and" connective, which connects two statements P and Q, and will be true only when P and Q are both true. The notation for "P and Q" is $P \wedge Q$. Here is the truth table:

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

This certainly make sense as far as our intuition goes. If I try to state that "I am hungry and I am thirsty", it better be true that "I am hungry" AND "I am thirsty". I wouldn't be able to say "I am hungry and I am thirsty" if I was only hungry but not thirsty.

Or

We introduce that "or" connective in terms of the english word "or", except there are two interpretations of this word, depending on the context.

Example 1: "Let's go see a movie. Do you want to watch *Inception* or *Spiderman*?"

Example 2: "If you want to fill up a bucket with water, take it out in the rain, or fill it up with a tap."

The word "or" in Example 1 is what we call the "exclusive or", which means "one of these two things, but not both" (because when we're going out to see a movie, we're not going to watch two at once. We will pick one, and we will not pick the other). In Example 2, we see the "inclusive or", which means "either of these things, but both would be okay too." (because, of course, if you brought a bucket out in the rain AND held it under a tap, you would still fill it. It might be more than enough, but it doesn't change that it works to do both). To make sure we define things well and unambiguously, logicians and mathematicians have agreed that when we see or say the word "or", it is the INCLUSIVE or, as in Example 2. We denote this "or" connective by $\vee$ and we thus get the corresponding truth table:

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

It makes sense, here, that the only way that $P \vee Q$ could be false is if both P and Q were false.

As you can imagine, the exclusive or, discussed above, would have a similar truth table, except there would be an F in the first row instead of the T in the above table.

Here's an interesting result: The statements $\sim (P \wedge Q)$ and $\sim P \ \vee \sim Q$ have the same truth values, and are thus logically equivalent. This probably make sense, intuitively, because if I don't have two things, it means I don't have one of them, or maybe I don't have the other, or maybe I just don't have either. This can be proven by building the truth tables from the previous truth tables:

P	Q	$\sim P$	$\sim Q$	$P \wedge Q$	$\sim P \ \vee \sim Q$	$\sim (P \wedge Q)$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

In the table above, we set up the first two columns as the possible combinations of truth values of P and of Q. Then the next three columns are intermediate steps to get us to the last two columns, which contain the final expressions. We see that these last two columns are the same, which tells us what we were trying to prove (that $\sim (P \wedge Q)$ and $\sim P \ \vee \sim Q$ have the same truth values). There is another similar rule, which says that $\sim (P \vee Q)$ and $\sim P \ \wedge \sim Q$ have the same truth values, which you can prove yourself.