Grade 7/8 Math Circles

Fall 2012

Factors and Primes

Factors

Definition: A factor of a number is a whole number that divides evenly into the other number. So, when you divide by the factor the remainder is zero. Factors come in pairs. When you divide a number by one of its factors, the answer or quotient is the paired factor.

Example 1

a) Is 6 a factor of 42?

Yes, since 42 divided by 6 gives us an answer of 7 with no remainder.

b) Is 6 a factor of 26?

No, since 26 divided by 6 gives us an answer of 4 with a remainder of 2.

Exercise 1

List all of the factors of the following numbers:

1. 24: 1, 2, 3, 4, 6, 8, 12, 24

2. 12: 1, 2, 3, 4, 6, 12

3. 64: 1, 2, 4, 8, 16, 32, 64

4. 17: 1, 17

Question: Did you find anything special about the last two numbers?

- 64 has an odd number of factors.

  Each factor has a pair, but 8 pairs with itself, so it is only listed once. This means that 64 is a perfect square. Others are 1, 4, 9, 16, 25, 36, 49 and so on.

- 17 only has two factors, 1 and itself.

  This means that 17 is a prime number. The first few are 2, 3, 5, 7, 11, 13 and so on.

Definitions:

A prime number is a number that has only two factors: 1 and itself.

A composite number is a number with factors other than 1 and itself.

Question: Are there any even prime numbers?

- Yes, 2 is the only prime number.

Question: Is 1 a prime number or a composite number?

- By definition, it is neither composite nor prime.

Prime Factorization

One way that we can describe a number is by breaking them down into a product of their prime factors. This is called prime factorization. By definition, the prime factorization of a prime number is the number itself, and the prime factorization of 1 is 1.
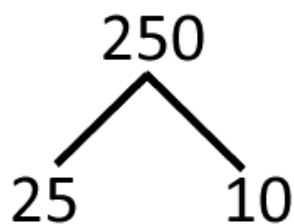
Unique Factorization Theorem

Every integer greater than 1 can be represented in exactly one way as the product of one or more prime numbers.
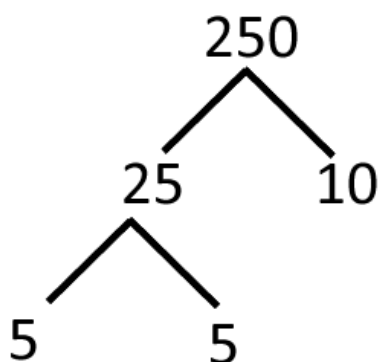
Example 2

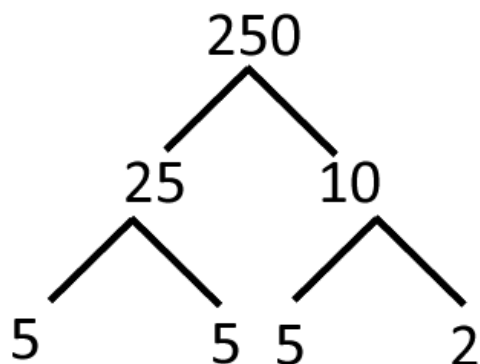Let's find the prime factorization of 250 using a factor tree.

Let's first think of a pair of factors of 250. If we use 1 and 250, we won't make any progress, so we will choose another pair. Let's choose 10 and 25.

$$250$$
$$25 \qquad 10$$

Now let's look at the first factor, 25. Is 25 prime? No, so we have to find its prime factorization. What is a pair of factors of 25? We will use 5 and 5. Now, 5 is a prime number, so we are finished with this part.

$$250$$
$$25 \qquad 10$$
$$5 \qquad 5$$

Next we will work on the right side. Is 10 prime? No, so we have to find its prime factor. We will factor it using 5 and 2. And since both 5 and 2 are prime, that means that we are done.

$$250$$
$$25 \qquad 10$$
$$5 \qquad 5 \quad 5 \qquad 2$$

So, we have found the prime factorization of 250, which we will write as a product of these prime numbers.

$$250 = 5 \times 5 \times 5 \times 2 = 2 \times 5^3$$

Divisibility Tests

The rules listed below are used to determine whether a number has any of the following numbers as a factor. Also, note that 0 is divisible by all positive numbers.

Divisible by?	Divisibility Test	Example
2	The ones digit is even.	124 is divisible by 2 since 4 is an even number.
3	The sum of all the digits is divisible by 3.	4651563 is divisible by 3 since $4 + 6 + 5 + 1 + 5 + 6 + 3 = 30$, which is divisible by 3.
4	The last two digits are divisible by 4.	258048 is divisible by 4 since 48 is divisible by 4.
5	The ones digit is 0 or 5.	215640 is divisible by 5 since the ones digit is 0.
6	The number is divisible by 2 and 3.	2766 is divisible by 6 since 6 is even so it is divisible by 2 and $2 + 7 + 6 + 6 = 21$ which is divisible by 3.
7	Subtract twice the ones digit from the remaining digits of the number until you get a small number that is divisible by 7.	854 is divisible by 7 since $4 \times 2 = 8$, and $85 - 8 = 77$, which is divisible by 7.
8	The last three digits are divisible by 8.	4512016 is divisible by 8 since 016 is divisible by 8.
9	The sum of all the digits is divisible by 9.	383076 is divisible by 9 since $3+8+3+0+7+6 = 27$ which is divisible by 9.
10	The ones digits is 0.	542160 is divisible by 10 since the ones digit is 0.
11	The alternating sum of the digits is divisible by 11.	3267 is divisible by 11 since $3 - 2 + 6 - 7 = 0$, which is divisible by 11.

Greatest Common Divisor

Definition: The greatest common divisor or gcd of two or more numbers is the largest number that is a factor of each of the numbers.

We are going to look at two ways to find the gcd.

Method 1: The first way is to find all of the pairs of factors of each of the numbers and then find the largest factor that is a factor of all of the numbers.

Example 3: Find the greatest common divisor of 64 and 56.

List the factors of:

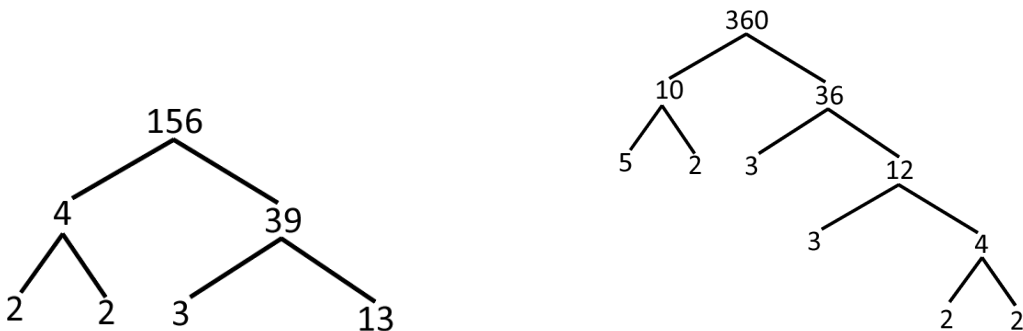- 64: 1, 2, 4, 8, 16, 32, 64

- 56: 1, 2, 4, 7, 8, 14, 28, 56

The factors that 64 and 56 have in common are: 1, 2, 4, 8

Of these factors, 8 is the largest, so the gcd of 64 and 56 is 8.

Method 2: The second way of finding the gcd is to:

1. Find the prime factorization of both numbers.

2. Look at the prime factors and multiply the ones that they have in common.

Find the greatest common divisor of 156 and 360. (Start with the factor trees.)

So the prime factorization of 156 is: $156 = 2 \times 2 \times 3 \times 13 = 2^2 \times 3 \times 13$

And the prime factorization of 360 is: $360 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 = 2^3 \times 3^2 \times 5$

So the common prime factors are: $2 \times 2 \times 3 = 2^2 \times 3 = 4 \times 3 = 12$

Therefore the greatest common divisor of 156 and 360 is: 12

Problem Set

1. Find the factors of the following numbers (use the divisibility tests when applicable)

   (a) 98

   (b) 5426

   (c) 97

   (d) 169

2. Draw the factor trees for the following numbers:

   (a) 2459

   (b) 456

   (c) 123

   (d) 54320

3. Mr. Z wants to split his class of 36 students into equal groups for a group project. List all of the possibilities of groups that Mr. Z can divide his class into so that no students are left without a group.

4. Draw two different factor trees for the number 180. (ie Start with two different pairs of factors.) What do you notice about their prime factorizations?

5. Find the gcd of the following numbers:

   (a) 542, 562

   (b) 700, 28

   (c) 123, 1025

   (d) 378, 294

   (e) 54320, 5240, 2080

6. The volume of a box is $1925cm^3$. What are the different possible dimensions of the box?

   (Note: the volume of a rectangular box is $length \times width \times height$.)

7. Michael Jordan has 72 basketballs, 84 baseballs and 48 golf balls and he wants to group them together so that he can donate them to schools in his hometown. What is the largest number of identical groups he can put together without having any balls left over?

8. Using the divisibility tests, determine which numbers (2 to 11) are factors of the following:

   (a) 926

   (b) 1420

   (c) 7848

   (d) 22176

   (e) 39208

   (f) 55440

9. What is the smallest number that you must multiply 48 by so that the product is divisible by 45?

10. The eight digit number 1234□678 is divisible by 11. What is the digit □?

11. What is the smallest prime number greater than 200?

12. The four digit number 43□□ is divisible by 3, 4 and 5. What are the last two digits?

13. The product of three different positive integers is 144. What is the maximum possible sum of these three integers?

14. What is the smallest number that you must multiply 6750 by in order to get a perfect square (which is greater than 6750)?

15. The gcd of two numbers is 850 and neither is divisible by the other. What is the smallest that these two numbers could be?

16. * Two campers have flashlights that blink every few seconds. The first camper's flashlight blinks every 4 seconds and the second's blinks every 6 seconds. If they turn on their flashlights at the same time, how many times will they flash simultaneously in one minute?

17. * James is buying hamburgers and hamburger buns for his son's 6th birthday party. Hamburgers come in packs of 24, but buns come in packs of 16. How many packs of each will James have to buy in order to have no hamburgers or buns left over?

18. ** Two numbers are *co-prime* if they do not have any prime factors in common. List all possible numbers that are co-primes with the following numbers that are < 50:

   (a) 390

   (b) 210

   (c) 49335