Math Circles: Combinatorics II

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1 Counting via 1-1 correspondences

Combinatorics is the branch of mathematics that deals with counting objects. We might thus begin by asking how one can formalize the notion of counting. For example, if we have a set of four elements, say $A = \{1, 2, 3, 4\}$, and another set with three elements, such as $\{a, b, c\}$, in what sense can we say (formally) that the two sets have different numbers of elements? The following definition gives one possible approach.

Definition 1.1. We say that two sets A and B have the same size if there exists a function f from A to B which gives a 1-1 correspondence between A and B.

Remark 1.2. A function f from A to B is denoted by the notation $f\colon A \to B$.

Exercise 1.3. Let A, B, and C be three sets. Show that:

1. A has the same size as A.
2. If A has the same size as B, then B has the same size as A.
3. If A has the same size as B, and B has the same size as C, then A has the same size as C.

We can now show that $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$ do not have the same size. The most straightforward method is to write down all possible functions from A to B and show that none of them is a 1-1 correspondence. This method is quite easy but also tedious. We will instead use a different approach which is more insightful and generalizes easily. We begin with the following easy result.

Lemma 1.4. *The sets $\{1, 2\}$ and $\{a\}$ do not have the same size.*

Proof. There is only one function f from $\{1, 2\}$ to $\{a\}$, given by $f(1) = f(2) = a$, and this function is not a 1-1 correspondence because $f(1) = f(2)$ but $1 \neq 2$. □

Now we add an element:

Lemma 1.5. *The sets $\{1, 2, 3\}$ and $\{a, b\}$ do not have the same size.*

Proof. Suppose $f\colon \{1, 2, 3\} \to \{a, b\}$ were a 1-1 correspondence. Without loss of generality, let $f(3) = b$ (if $f(3) = a$ then rename a to b and vice-versa). Then f restricted to $\{1, 2\}$ is a 1-1 correspondence between $\{1, 2\}$ and $\{a\}$, contradicting Lemma 1.4. □

Exercise 1.6. Prove that $\{1, 2, 3, 4\}$ and $\{a, b, c\}$ do not have the same size.

Once the set of natural numbers has been constructed, using the *principle of mathematical induction* one can generalize the above chain of reasoning to arbitrary finite sets. We will not give the construction of natural numbers here, since it is beyond the scope of this discussion. But the resulting theorem concerning the sizes of finite sets is a very well-known theorem called the *Pigeonhole Principle*, and we state it here:

Theorem 1.7 (Pigeonhole Principle)**.** *Let m and n be natural numbers. Let A be a finite set of n elements, and B be a finite set of m elements. Then A and B have the same size if and only if $n = m$.*

2 Infinite sets

We naturally ask if the idea of 1-1 correspondences can be used to count infinite sets. The answer is yes, as we illustrate with some examples. Let $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ denote the set of *natural numbers*, i.e. positive integers.

Proposition 2.1. *Let n be a natural number. The sets $\mathbb{N}$ and $\{1, 2, 3, \ldots, n\}$ do not have the same size.*

Proof. We use the principle of mathematical induction. For $n = 1$ the theorem is easy to prove: there is only one function $f\colon \mathbb{N} \to \{1\}$, and it is not a 1-1 correspondence because $f(1) = f(2) = 1$ but $1 \neq 2$. Now suppose the theorem holds for $n \in \mathbb{N}$. Let $f\colon \mathbb{N} \to \{1, 2, 3, \ldots, n + 1\}$ be any function. Assume that f is a 1-1 correspondence. Then there exists a unique $x \in \mathbb{N}$ such that $f(x) = n + 1$. Let $\mathbb{N} \setminus \{x\}$ (a.k.a. "$\mathbb{N}$ minus $\{x\}$") denote the set $\mathbb{N}$ with the element x deleted. Then f restricted to $\mathbb{N} \setminus \{x\}$ is a 1-1 correspondence between $\mathbb{N} \setminus \{x\}$ and $\{1, 2, 3, \ldots, n\}$, so these two sets have the same size. But $\mathbb{N}$ and $\mathbb{N} \setminus \{x\}$ also have the same size (why?), so by Exercise 1.3, the sets $\mathbb{N}$ and $\{1, 2, 3, \ldots, n\}$ have the same size, which is a contradiction. $\square$

Proposition 2.1 is not a surprising result. However, other similar results concerning the sizes of infinite sets are more surprising:

Proposition 2.2. *Let $2\mathbb{N} = \{2, 4, 6, 8, \ldots\}$ denote the set of even integers. Then $\mathbb{N}$ and $2\mathbb{N}$ have the same size.*

Proof. Define $f\colon \mathbb{N} \to 2\mathbb{N}$ by $f(x) = 2x$. We will prove that f is a 1-1 correspondence.

For every natural number x, the natural number $2x$ is even (by definition of even), so $f(x) \in 2\mathbb{N}$ and thus f is well-defined. Furthermore, every natural number n is divisible by 2, and hence of the form $n = 2k$ for some $k \in \mathbb{N}$, implying that $f(k) = n$. This result shows that f is onto. Finally, if $f(x) = f(y)$, then $2x = 2y$. By cancelling the 2's, we obtain $x = y$. Therefore f is 1-1. $\square$

A similar argument shows that the integers $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ and the natural numbers $\mathbb{N}$ have the same size:

Proposition 2.3. *$\mathbb{N}$ and $\mathbb{Z}$ have the same size.*

Proof. Define $f\colon \mathbb{N} \to \mathbb{Z}$ by the piecewise rule

$$f(x) = \begin{cases} \frac{x}{2} & x \text{ even}, \\ \frac{-x-1}{2} & x \text{ odd}. \end{cases}$$

One can show (exercise) that f is a 1-1 correspondence between $\mathbb{N}$ and $\mathbb{Z}$. $\square$

Perhaps more surprising is that even the set $\mathbb{Q}$ of *rational* numbers has the same size as $\mathbb{N}$:

Theorem 2.4. *$\mathbb{N}$ and $\mathbb{Q}$ have the same size.*

To prove this theorem, arrange the rational numbers in a 2-dimensional half-plane array and count outwards in a spiral starting from 0, as in Figure 1. This technique overcounts some rational numbers, since not every term in the array is in lowest form, but one can get around this problem by skipping over any rational number which has already been counted.

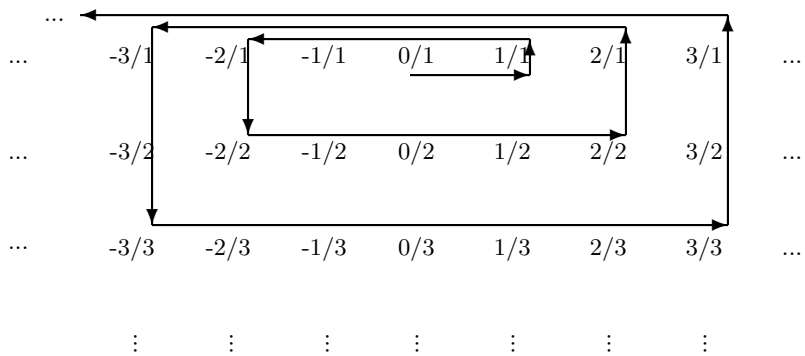

Figure 1: Counting the rational numbers.

Remark 2.5. Under the theory of cardinal arithmetic, Propositions 2.2 and 2.3 might be interpreted as saying that $\infty + \infty = \infty$ or that $2 \cdot \infty = \infty$, and Theorem 2.4 can be taken to mean $\infty \cdot \infty = \infty$.

3 Uncountable sets

We have seen that all the number systems up to $\mathbb{Q}$ have the same size as the natural numbers. Does this mean that *all* infinite sets have the same size? Again, surprisingly, the answer is no. It turns out that the set of *real numbers* does not have the same size as the set of natural numbers. The first person to prove this fact was Georg Cantor, and his proof, nowadays referred to as *Cantor's diagonal argument*, is a landmark result in the development of modern mathematics.

In order to discuss real numbers we do need a minimally accurate working definition of real numbers. The following definition suffices for our purposes.

Definition 3.1. A real number is defined to be an integer followed by an infinite sequence of decimal digits, where each digit is taken from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. The set of all real numbers is denoted $\mathbb{R}$.

Note that, although the definition requires an infinite sequence of decimal digits, a finite sequence can in fact be represented by taking an infinite tail of zeros. The definition is also incomplete in that there exist pairs of different sequences which yield the same number (most notoriously, $0.9999\ldots = 1.000\ldots$), but this issue does not affect the analysis which follows.

Theorem 3.2. *$\mathbb{N}$ and $\mathbb{R}$ do not have the same size.*

Proof. Suppose that $f\colon \mathbb{N} \to \mathbb{R}$ is any function. We will show that f is not a 1-1 correspondence, by constructing a real number which does not equal $f(n)$ for any n.

Write

$$\begin{aligned} f(1) &= x_1.y_{11}y_{12}y_{13}\dots \\ f(2) &= x_2.y_{21}y_{22}y_{23}\dots \\ f(3) &= x_3.y_{31}y_{32}y_{33}\dots \\ f(4) &= x_4.y_{41}y_{42}y_{43}\dots \\ &\vdots \end{aligned}$$

where $x_n \in \mathbb{Z}$ denotes the integer part of $f(n)$, and $y_{nm} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ denotes the m-th decimal digit in the decimal part of $f(n)$. Construct a new real number

$$r = 0.b_1b_2b_3b_4\dots$$

using the following method: for each $n \in \mathbb{N}$, choose the digit b_n to be any digit *not* equal to 0, 9, or y_{nn}. We claim that $f(n) \neq r$ for all $n \in \mathbb{N}$. Suppose to the contrary that $f(n) = r$. Then y_{nn} is the n-th digit of $f(n)$, and b_n is the n-th digit of r, so $y_{nn} = b_n$. But this contradicts the definition of b_n.

(The reason why we take $b_n \neq 0$ or 9 is to avoid problems such as $0.9999\dots = 1.000\dots$ where two different decimal sequences yield the same real number.) $\square$

In cardinal arithmetic, the size of $\mathbb{R}$ can be expressed as 2^∞, and Theorem 3.2 amounts to the statement that $2^\infty > \infty$. Any infinite set of size larger than $\mathbb{N}$ is called an *uncountable* set. The famous *continuum hypothesis* of Cantor asserts that there is no set larger than $\mathbb{N}$ and smaller than $\mathbb{R}$. Paul Cohen showed in 1963 that the continuum hypothesis can neither be proved nor disproved under the standard axioms of set theory.