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Mathematics and Computing

Grade 6 Math Circles
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Introduction to Number Theory

Number Theory

Many mathematicians have discovered certain numbers that have unique characteristics and through study have found patterns with these numbers (though there are still many answers and patterns that have not been found). Number theory involves this type of study and is a branch of pure mathematics. Today we will investigate certain types of numbers involving prime, perfect, abundant and deficient numbers.



Prime Numbers

Prime numbers are numbers that can only be divided evenly by one and itself (*eg*: 7 is a prime number since it only has divisors of 1 and 7).

Finding Prime Numbers

A famous mathematician named Eratosthenes often worked with numbers and patterns within numbers. We will look at one of his most popular methods that helps us find prime numbers. This is called the Sieve of Eratosthenes.

Using your 15×15 board (with numbers 1-225) provided in class follow the steps:

1. Cross out 1 (*Note*: 1 is considered a special case when dealing with prime numbers)
2. Circle 2
3. Cross out all the multiples of 2
4. Circle the next none crossed out number (3)
5. Cross out all the multiples of 3
6. Continue this way until every number in the first row (1-15) is either circled or crossed out
7. Circle all the numbers that are left and are not crossed out.
8. All the numbers that are circled are prime numbers!

Prime Factorization

Every number can be expressed as a product of prime numbers. This expression is known as *prime factorization* (*eg*: 3×5 is the prime factorization of 15).

Example Set 1

1. Find the prime factorization of the following numbers:

| | |
|---------|-----------------|
| (a) 39 | (d) 498 |
| (b) 8 | (e) 999 |
| (c) 100 | (f) 2, 268, 000 |

Sum of Divisors

Sometimes, it will be useful to find the sum of all the divisors. With small numbers, this is not hard to do, but as the numbers increase, this task becomes much more tedious. There is in fact, a quicker way to do this!

Looking at $100 = 2^2 \times 5^2$, it can be shown that all the divisors of 100 must be a combination of 2's (up to two 2's) and 5's (up to 2 5's).

| | 2^0 | 2^1 | 2^2 | Row Sum |
|-------------------|-------|-------|-------|----------------|
| 5^0 | 1 | 2 | 4 | 7 |
| 5^1 | 5 | 10 | 20 | 35 |
| 5^2 | 25 | 50 | 100 | 175 |
| Column Sum | 31 | 62 | 124 | 217 |

Some Things to Note:

- The sum of the first column is 31, and the other column sums are just being multiplied by 2
- The sum of the first row is 7, and the other row sums are just being multiplied by 5
- The sum of all the column sums is the same as the sum of all the row sums
- Every divisor of 100 is in the center part of the table.
- 217 is the sum of all the divisors of 100

Though the table is useful for producing the divisors of 100, an easier way to find the sum of the divisors is multiplying the sum of the first row by the sum of the second row. This can be shown with the following algebra:

$$(\text{Sum of First Row}) = 2^0 + 2^1 + 2^2 = 7$$

$$(\text{Sum of First Column}) = 5^0 + 5^1 + 5^2 = 31$$

$$(\text{Sum of Divisors}) = (\text{Sum of First Row}) \times (\text{Sum of First Column})$$

$$(\text{Sum of Divisors}) = 7 \times 31 = 217$$

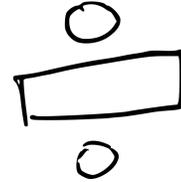
Example Set 2

- Using the formula above, find the sum of the divisors of all the numbers in *Example Set 1* (excluding 100).

Proper Divisors

A *proper divisor* is a positive divisor of a number, excluding the number itself (i.e. every number that can be evenly divided into n , excluding n)

Once we have found the sum of the divisors of a number, all we have to do is subtract the number itself to find the sum of all the proper divisors (*eg*: $217-100=117$ is the sum of the proper divisors of 100).



Example Set 3

- Find all the proper divisors of the following numbers:

| | |
|---------|---------|
| (a) 12 | (e) 6 |
| (b) 7 | (f) 110 |
| (c) 30 | (g) 93 |
| (d) 104 | (h) 28 |
- Find the sum of all the proper divisors of the numbers in *Example Set 1* (excluding 100).

Abundant Numbers

An *abundant number* is a number whose sum of proper divisors is more than the number itself.

For example, 12 is an abundant number since the proper divisors are 1,2,3,4 and 6 and

$$1 + 2 + 3 + 4 + 6 = 16 > 12$$

Deficient Numbers

A *deficient number* is a number whose sum of proper divisors is less than the number itself

For example, 110 is a deficient number since the proper divisors are 1,2,5,10,11,22 and 55 and

$$1 + 2 + 5 + 10 + 11 + 22 + 55 = 106 < 110$$

Perfect Numbers

A *perfect number* is a number whose sum of proper divisors is equal to the number itself

For example, 6 is a deficient number since the proper divisors are 1, 2 and 3 and $1 + 2 + 3 = 6$ (which is the number we started with)



Mathematicians have found that there are not many perfect numbers. In fact, the fifth perfect number is 8 digits long (and the thirty-eighth perfect number is roughly 4 million digits long)!

Did you Know?

The sum of all the reciprocals of all the divisors of any perfect number (including the number itself) is always 2!

For example, looking at 6 we have $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 2$

Finding Perfect Numbers

Although there is not many perfect numbers, the famous mathematician Euclid did discover a method for finding perfect numbers.

Firstly, you must start with 1 (2^0) and keep adding powers of 2 until the sum is a *prime number* (a number that is only divisible by one and itself).

Once the prime number is attained, the perfect number is found by multiplying the sum and the last power of 2 together.

Fill in the Following Chart:

| Powers of 2 | Sum | Prime | Calculation | Perfect Number |
|-------------|-------|-------|--------------|----------------|
| $1 + 2$ | $= 3$ | YES | 2×3 | $= 6$ |
| | | | | |
| | | | | |
| | | | | |

Example Set 4

1. Determine whether each number in *Example Set 1* is abundant, deficient, or perfect.