Centre for Education in Mathematics and Computing
Math Circles - February 16, 2011
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Probability Problems Set 2

1. **Pairwise Best - Worst Paradox** You are to play a game in which there are three urns. In each urn there are some balls and each ball has a number on it. You choose an urn and your opponent chooses an urn from the remaining ones. You each choose one ball at random from your urn. Who ever draws the ball with the higher number wins. In urn A, there is one ball numbered 3. In urn B, there are 56 balls numbered 2, 22 balls numbered 4, and 22 balls numbered 6. In urn C, there are 51 balls numbered 1 and 49 balls numbered 5. Which urn would you select? Which urn would be your worst choice? Now, a third player comes along. Each of you will get an urn and each will draw one ball. The player with the highest number will win. Now which urn do you choose?

2. In the game *Risk*, there are times when one player tosses three dice and another player tosses two dice. The person who tosses the largest number "wins". What is the probability distribution for the largest number tossed by the player who tosses three dice? What is the probability that the player who tosses the three dice wins?

3. For Lotto 6/49, the prizes and probabilities are given in the table below.

Match	Prize	Probability
6 of 6	Share of 80.50% of Pools Fund	1/13,983,816
5 of 6 + Bonus	Share of 5.75% of Pools Fund	1/2,330,636
5 of 6 (no bonus)	Share of 4.75% of Pools Fund	1/55,491
4 of 6	Share of 9.00% of Pools Fund	1/1,032
3 of 6	$10	1/57
2 of 6 + Bonus	$5	1/81

   Suppose you buy 5 different tickets for a draw. What is the probability that you win the grand prize? If you use this strategy every week for a year, what is the probability that you will win the grand prize at least once? How many tickets would you need to buy every week for one year so that the probability you win the grand prize at least once is 10%?

4. You want to find someone whose birthday matches yours, so you approach people on the street asking each one for their birthday. Ignoring leap year babies, if we assume that each person you approach has probability 1/365 of sharing your birthday, *on average* how many people would you have to approach to find someone with the same birthday as you?

5. Eight red blocks and four blue blocks are arranged at random in a row. What is the probability that no two blue blocks are side by side?

6. You will be presented with 10 offers to buy your car. You, quite naturally, want to choose the best offer. However, you see the offers one at a time, and if you refuse any offer, you do not get another chance at that offer. You know if you choose an offer at random, there is probability 1/10 of choosing the best offer. So you try the following strategy: Examine the first four offers but refuse each one, noting which of these is the best. Then for each of the other offers, refuse it if it is lower than the best one you saw in the first four, and accept it if it is better than the best one you saw in the first four. If you have not selected an offer by the time the last one comes in, you take the last one (by now you are desperate). What is the probability that by using this strategy you will select the best offer?

7. You want to find someone whose birthday matches yours, so you approach people on the street asking each one for their birthday. Ignoring leap year babies, if we assume that each person you approach has probability $1/365$ of sharing your birthday, *on average* how many people would you have to approach to find someone with the same birthday as you?

8. **Cereal Box Problem** A brand of breakfast cereal offers an action figure prize in each package. If there are six different action figures and there are equal numbers of each figure distributed at random, one per package, how many boxes would you expect to buy in order to have the complete set? You can assume there are a very large number of boxes of cereal.

9. A fair die is rolled n times and the value on the upper face is noted Let X be a random variable giving the number of distinct values in the n rolls (e.g. if, for $n = 8$ you observe $\{4,5,3,3,6,1,4,4\}$, then $X = 5$). Find $E(X)$. What happens to $E(X)$ as n goes to infinity?

10. (1993 Euclid Contest #10) In a sequence of p zeros and q ones, the *ith* term, t_i, is called a *change point* if $t_i \neq t_{i-1}$, for $i = 2, 3, 4 \ldots, p + q$. For example, the sequence $0, 1, 1, 0, 0, 1, 0, 1$ has $p = q = 4$, and five change points t_2, t_4, t_6, t_7, t_8. For all possible sequences of p zeros and q ones, with $p \leq q$, determine:

    (a) the minimum and maximum number of change points.

    (b) the average number of change points.

11. (1997 Euclid Contest #10) A group of n married couples arrives at a dinner party and is seated around a circular table. The distance between the members of a couple is defined to be the number of people sitting between them measured either clockwise or counter-clockwise, whichever gives the smaller result.

    (a) Considering all possible seating arrangements of the $2n$ people, what is the average distance between a particular couple A?

    (b) Considering all possible seating arrangements for the $2n$ people, what is the average number of couples, per arrangement, where both members of the couple are seated side-by-side?

    (c) Repeat the question if the $2n$ people are attending the theatre and are seated in a row. (The distance between members of a couple is defined here as the number of people sitting between them).

12. A sample of size n is chosen at random *with replacement* from the integers $1, 2, \ldots, N$. (In sampling *with* replacement, an item is drawn, we note which item it was, and it is then replaced. Note that not all integers need to be different under sampling with replacement). Find the average number of *different* integers in the sample. What happens if $n \ll N$?

13. A random graph with n vertices is generated by connecting pairs of vertices at random. Each of the possible $\binom{n}{2}$ possible edges is inserted with probability p, independently of other edges. Find the average number of triangles in the graph.

14. n people toss their hats in the air and the hats are picked up randomly. Each person who gets his or her own hat leaves and the remaining people toss the hats again. The process continues until every person has received his or her own hat again. Find the average number of rounds of tosses required.