Solutions

Problem 1

Find the midpoint of line AC and line BD:

\[ M_{AC} = \left( \frac{2b + 2c}{2}, \frac{2d}{2} \right) \quad \text{and} \quad M_{BD} = \left( \frac{2b + 2c}{2}, \frac{2d}{2} \right) = (b + c, d) \]

Therefore, since the midpoints of each line are the same point, it is the point of intersection, and so the diagonals of a parallelogram bisect each other.

Problem 2

\[ m_{PQ} = \frac{e}{(b + c) - b} = \frac{e}{c} \quad \text{and} \quad m_{SR} = \frac{(e + f) - f}{(c + d) - (b + c)} = \frac{e}{c} \]

\[ m_{QR} = \frac{(e + f) - e}{(c + d) - (b + c)} = \frac{f}{d - b} \quad \text{and} \quad m_{PS} = \frac{f}{d - b} \]

\[ \therefore m_{PQ} = m_{SR} \quad \text{and} \quad m_{QR} = m_{PS}, \text{ so } PQRS \text{ is a parallelogram.} \]
Problem 3

\[
m_{BC} = \frac{2d}{2c-2b} = \frac{d}{c-b} \quad \text{and} \quad BC = \sqrt{(2c-2b)^2 + (2d)^2} = 2\sqrt{(c-b)^2 + d^2}
\]

\[
m_{ED} = \frac{d}{c-b} \quad \text{and} \quad ED = \sqrt{(c-b)^2 + d^2} = \frac{1}{2}BC
\]

Therefore \( m_{BC} = m_{ED} \) and \( ED = \frac{1}{2}BC \) so the line segment joining the midpoints of two sides of the triangle is parallel to the third side and one-half the length of the third side.

Problem 4

\[
A_{\triangle ABC} = A_{\text{rectangle}} - A_{\text{outer triangles}}
\]
\[
A_{\text{rectangle}} = A_{\triangle ADEF} = (12 - (-2)) \times (5 - (-11)) = 14 \times 16 = 224
\]
\[
A_{\text{outer triangles}} = A_{\triangle ADC} + A_{\triangle CEB} + A_{\triangle ABF} = \frac{1}{2}[14(14) + 2(2) + 16(12)] = 196
\]
\[
\therefore A_{\triangle ABC} = 224 - 196 = 28
\]
Verification using formula for area of a triangle:
\[
A_{\triangle ABC} = \frac{1}{2} \left| 10(-11) + (-2)(3) + 12(5) - 5(-2) - (-11)(12) - 3(10) \right|
\]
\[
= \frac{1}{2} \left| -110 - 6 + 60 + 10 + 132 - 30 \right|
\]
\[
= 28
\]

**Problem 5**

\[
\text{Area} = \frac{1}{2} \left| x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_3y_2 - x_2y_1 \right|
\]
\[
= \frac{1}{2} \left| (-12)(-3) + (-4)(-8) + (6)(1) - (-12)(-8) - (6)(-3) - (-4)(1) \right|
\]
\[
= \frac{1}{2} \left| 36 + 32 + 6 - 96 + 18 + 4 \right|
\]
\[
= 0
\]

The triangle does not have an area because they all lie on the same line.

**Problem 6**

\[
\text{Area} = \frac{1}{2} \left| (4p) + (6)(6) + (0)(3) - (4)(6) - (0)p - (6)(3) \right|
\]
\[
7 = \frac{1}{2} \left| 4p + 36 + 0 - 24 - 0 - 18 \right|
\]
\[
14 = |4p - 6|
\]
\[
7 = |2p - 3|
\]

Since we have an absolute value, we have two answers.
\[
7 = 2p - 3 \quad \quad \quad -7 = 2p - 3
\]
\[
p = 5 \quad \quad \quad p = -2
\]

**Problem 7**

Find point of intersection of \( BD \) and \( AC \):

Equation of line \( BD \):
\[
y = -\frac{8}{p}x + 8
\]
\[
x = (8 - y)\frac{p}{8}
\]

Equating lines \( BD \) and \( AC \):
\[
(8 - y)\frac{p}{8} = \frac{p}{10}y
\]
\[
10(8 - y) = 8y
\]
\[
80 = 18y
\]

Equation of line \( AC \):
\[
y = \frac{40}{9}
\]
\[
y = \frac{10}{p}x
\]
\[
x = \frac{p}{10}y
\]
Problem 8

\[ d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} \]
\[ = \frac{|(2)(-3) + (-7)(5) + 1|}{\sqrt{2^2 + (-7)^2}} \]
\[ = \frac{|-6 - 35 + 1|}{\sqrt{4 + 49}} \]
\[ = \frac{40}{\sqrt{53}} \]

Problem 9

The slope the line is \( \frac{1}{3} \), so the slope of the perpendicular is \(-3\).

\[-3 = \frac{1}{3} x - \frac{2}{3} - (-6) \]
\[-9(x - 2) = x - 2 + 18 \]
\[-9x + 18 = x + 16 \]
\[10x = 2 \]
\[x = \frac{1}{5} \Rightarrow y = -\frac{3}{5} \]
Problem 10

Method 1:
By plotting the triangle, we can locate D.

After that, it is clear that the length of $AD$ is 4.

Method 2:
The slope of line $BC$ is

$$m_{BC} = \frac{1 - 1}{8 - 1} = 0$$

Since the slope of $BC$ is 0, we know that any line perpendicular to BC is vertical. Thus $D = (6, 1)$, and the length of $AD$ is

$$\sqrt{(6 - 6)^2 + (5 - 1)^2} = 4$$

Method 3:
Using the formula for distance from a point $(6, 5)$ to a line $y - 1 = 0$, we have

$$d = \frac{|(0)(6) + (1)(5) + (-1)|}{\sqrt{0^2 + a^2}}$$

$$= \frac{|5 - 1|}{1}$$

$$= 4$$

Method 4:
Using the length of $BC$ is 7. The area of the triangle is

$$\text{Area} = \frac{1}{2} |(6)(1) + (1)(1) + (8)(5) - (6)(1) - (8)(1) - (1)(5)|$$

$$= \frac{1}{2} |6 + 1 + 40 - 6 - 8 - 5|$$

$$\frac{1}{2} bh = \frac{1}{2} (28)$$

$$7h = 28$$

$$h = 4$$
Problem 11

Let \( P = (p, 2p + 3) \) and \( Q = (q, -q + 2) \), The midpoint would be \( M = \left( \frac{p + q}{2}, \frac{(2p + 3) + (-q + 2)}{2} \right) \).

\[
\begin{align*}
\frac{p + q}{2} &= 2 \\
p + q &= 4
\end{align*}
\]

\[
\begin{align*}
\frac{(2p + 3) + (-q + 2)}{2} &= 5 \\
2p - q + 5 &= 10 \\
2p - q &= 5
\end{align*}
\]

Solving these two equations, we have

\[
(p + q) + (2p - q) = 4 + 5
\]

\[
3p = 9
\]

\[
p = 3 \Rightarrow q = 1
\]

\[
\therefore P = (3, 9) \text{ and } Q = (1, 1)
\]