Answers to Lecture 1 Problems:

1. The primes from 197 to 250 are: 197, 199, 211, 223, 227, 229, 233, 239, 241

2. (a) \[ 
\begin{align*}
\Omega(2) &= 1 \\
\Omega(3) &= 1 \\
\Omega(4) &= 2 \\
\Omega(5) &= 1 \\
\Omega(6) &= 2 \\
\Omega(7) &= 1 \\
\Omega(8) &= 3 \\
\Omega(9) &= 2 \\
\Omega(10) &= 2 \\
\Omega(11) &= 1 \\
\Omega(12) &= 3 \\
\Omega(13) &= 1 \\
\Omega(14) &= 2 \\
\Omega(15) &= 2 \\
\Omega(16) &= 4 \\
\Omega(17) &= 1 \\
\Omega(18) &= 3 \\
\Omega(19) &= 1 \\
\Omega(20) &= 3 \\
\end{align*}
\]

(b) \( \Omega(100) = 4, \Omega(400) = 6, \Omega(800) = 7 \)

(c) Let \( m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} \) and \( n = p_1^{\beta_1} p_2^{\beta_2} \cdots p_n^{\beta_n} \). Then \( \Omega(m) = \alpha_1 + \cdots + \alpha_n \) and \( \Omega(n) = \beta_1 + \cdots + \beta_n \). Also,

\[
m n = p_1^{\alpha_1 + \beta_1} p_2^{\alpha_2 + \beta_2} \cdots p_n^{\alpha_n + \beta_n}
\]

so

\[
\Omega(mn) = \alpha_1 + \beta_1 + \cdots + (\alpha_n + \beta_n) \\
\quad = \alpha_1 + \cdots + \alpha_n + \beta_1 + \cdots + \beta_n \\
\quad = \Omega(m) + \Omega(n)
\]

A function with this property is said to be **completely additive**.

(d)

\[
\begin{align*}
\Omega(210000) &= \Omega(210 \times 1000) = \Omega(210) + \Omega(1000) \\
&= [\Omega(21) + \Omega(10)] + [\Omega(10) + \Omega(100)] \\
&= [2 + 2] + [2 + 4] = 10
\end{align*}
\]

(e) \( \Omega(20!) = \sum_{i=2}^{20} \Omega(i) = 1 + 1 + 2 + 1 + 2 + 1 + 3 + 2 + 2 + 1 + 3 + 1 + 2 + 2 + 4 + 1 + 3 + 1 + 3 = 36 \)

3. The \( n \) consecutive numbers \((n+1)! + 2, (n+1)! + 3, \ldots, (n+1)! + (n + 1)\) are all composite. Hence, these form a prime gap of at least \( n \) numbers.

4. \( 22 = 11 + 11 = 5 + 17 = 3 + 19 \) is the smallest even number that can be written as a sum of two primes in three ways.

\( 34 = 17 + 17 = 23 + 11 = 29 + 5 = 31 + 3 \) is the smallest even number that can be written as a sum of two primes in four ways.

5. Pick any odd number \( x \) greater than 5. Then \( x = y + 3 \) where \( y \) is an even number greater than 2, so if Goldbach’s Conjecture is true, then \( y \) can be written as a sum of two primes, so \( x \) is written as a sum of three primes. Similarly, for any even number greater \( x \) than 5, we can write \( x = y + 2 \).